

# Robust controller synthesis for multivariable nonlinear systems with unmeasured disturbances

Emrod Elisante<sup>a</sup>, Gade Pandu Rangaiah<sup>a,\*</sup>, Srinivas Palanki<sup>b</sup>

<sup>a</sup>Department of Chemical & Biomolecular Engineering, National University of Singapore, Kent Ridge, Singapore 119260, Singapore

<sup>b</sup>Department of Chemical Engineering, Florida State University and Florida A & M University, 2525 Pottsdamer Street, Tallahassee, FL 32310-6046, USA

Received 24 September 2002; accepted 28 August 2003

## Abstract

This work concerns robust controller synthesis using the differential geometric concepts for minimum phase nonlinear systems with unmeasurable disturbances. A pseudo-linearization of the disturbance model at the input–output linearization stage is applied to yield a linear subsystem for controller design. Based on this linear model, a multi-loop controller framework is implemented, whereby  $\mu$ -synthesis is used to design off-line robust controller in the outer loop while state feedback is implemented in the inner loop. Through proper selection of weights, the outer robust controller is explicitly designed to address both uncertainty and disturbance rejection whereas the inner controller is used for on-line static state feedback. Numerical simulations are used to illustrate robustness of the controller for multi-input multi-output temperature control in two non-isothermal continuous stirred tank reactors in series.

© 2003 Elsevier Ltd. All rights reserved.

**Keywords:** Nonlinear control; Input–output linearization; Robust stability; Chemical processes; Multivariable systems; Process control

## 1. Introduction

During the past two decades there has been extensive research in the design of controllers for nonlinear processes using differential geometric techniques (Isidori, 1995; Nijmeijer and van der Schaft, 1990). One approach that has proved popular is the input/output (I/O) linearization approach. In this multi-loop design methodology, a coordinate transformation is utilized that results in a linear relation between the inputs and outputs (inner loop controller design). Then, an external linear controller is designed for this linear system to enforce desired performance and stability characteristics (outer loop controller design). The performance of this controller largely depends on the availability of an accurate model that leads to exact cancellation of nonlinear terms via the coordinate transformation (Henson and Seborg, 1997). However, due to uncertainty and disturbance this is seldom the case and often leads to poor performance.

In order to fully benefit from the power of the I/O linearization approach, the outer loop controller must be robust to uncertainty in a non-conservative manner and must

meet closed-loop objectives such as tracking, regulation and disturbance attenuation. One approach to design this outer loop linear controller is to use ad hoc tuning methods. For instance, Kravaris and Chung (1987) developed the globally linearizing controller (GLC) where the outer loop is a PID controller whose tuning parameters are chosen via extensive simulations. A second approach to handle uncertainty is to use Lyapunov's direct method. Early attempts to use this method (see, for instance, Kravaris and Palanki, 1988; Calvet and Arkun, 1992; Marino and Tomei, 1993; Christofides et al., 1996; Jiang and Mareels, 1997) typically led to controllers that require a large control effort which leads to implementation difficulties in the presence of actuator constraints. This problem of constraints has been resolved in El-Farra and Christofides (2001) where a class of robust nonlinear controllers was proposed that achieve robust stabilization without using unnecessarily large control action and accounting for input constraints. A third approach to handle uncertainty is to use linear robust control tools such as  $H_\infty$  and  $\mu$ -synthesis which explicitly account for uncertainty. Recently Kolavennu et al. (2000) applied a multi-model  $H_2/H_\infty$  approach to design a robust controller in the outer loop to account for parametric uncertainty for a SISO nonlinear system. This approach was later extended to

\* Corresponding author. Tel.: +65-68742187; fax: +65-67791936.

E-mail address: chegpr@nus.edu.sg (G.P. Rangaiah).

nonlinear multivariable systems with parametric uncertainty (Kolavennu et al., 2001; Palanki et al., 2003).

In this work, we follow the approximate linearization approach proposed in Palanki et al. (2003). However, the design methodology in this paper differs in several aspects in terms of the uncertainty considered as well as the technique used for handling uncertainty in the outer loop. In this work, we consider systems that are subject to *both* parametric uncertainty as well as unstructured uncertainty. The system is I/O linearized based on a nominal model. The effect of inexact cancellation of nonlinear terms is accounted by considering lumped additive uncertainty in the I/O linear model used for robust controller design via  $\mu$ -synthesis. The unmeasurable disturbance dynamics are included in the linear model for robust design through pseudo-linearization. The development of the controller design procedure is algorithmic in nature. Precise stability results (such as those shown in El-Farra and Christofides, 2001) are not considered in this work.

This paper is structured as follows. In Section 2, the control problem is formulated mathematically and the assumptions under which the controller is designed, are discussed. In Section 3, a robust controller design procedure is developed for the uncertain linear system that arises by applying a nominal I/O linearizing state feedback to the original multivariable nonlinear system. This controller synthesis procedure is applied to temperature control of a non-isothermal reaction taking place in two CSTRs in series, in Section 4. Finally, conclusions and recommendations for future work are discussed in Section 5.

## 2. Problem formulation

The following multivariable nonlinear system is considered:

$$\dot{x} = f(x) + g(x)u + w(x)d,$$

$$y = h(x), \quad (1)$$

where  $x \in \mathbb{R}^n$  is the vector of states;  $u \in \mathbb{R}^m$  is the vector of manipulated inputs;  $d \in \mathbb{R}^p$  is the disturbance inputs; and  $y \in \mathbb{R}^m$  is the vector of outputs.

There is uncertainty in  $f$  and  $w$  due to unmodeled dynamics and parametric uncertainty. The objective is to design a multi-loop controller in the I/O linearization framework as shown in Fig. 1. The controller design is developed under the following assumptions:

- (1) The function  $f$  is a smooth vector function,  $g$  consists of sufficiently smooth vector functions  $g_1, g_2, \dots, g_m$ ,  $w$  consists of sufficiently smooth vector functions  $w_1, w_2, \dots, w_m$ , and  $y$  is a smooth vector function.
- (2) The uncertainty in  $f$  and  $w$  can be represented in additive form as follows:

$$f = \tilde{f} + \Delta f,$$

$$w = \tilde{w} + \Delta w. \quad (2)$$

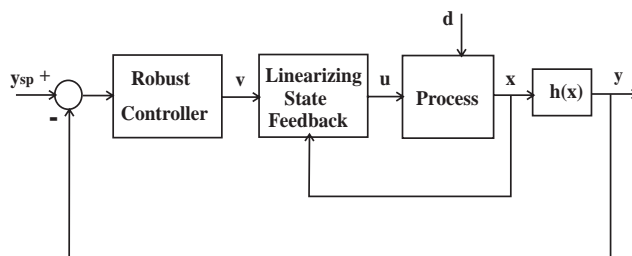


Fig. 1. Multi-loop robust controller design.

- (3) The relative degree of output  $y_i$  is  $r_i$ , with respect to each manipulated input,  $u_i$ , and  $\rho_i$ , with respect to each disturbance input,  $d_i$  is finite and does not change in the presence of uncertainty. The relative degree of the system with respect to the manipulated input is defined as

$$r = \sum_{i=1}^m r_i \quad (r \leq m).$$

- (4) The characteristic matrix of the nominal system, represented by

$$\mathcal{A}(x) = \begin{bmatrix} L_{g_1} L_{\tilde{f}}^{r_1-1} h_1(x) & \cdots & L_{g_m} L_{\tilde{f}}^{r_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_{\tilde{f}}^{r_m-1} h_m(x) & \cdots & L_{g_m} L_{\tilde{f}}^{r_m-1} h_m(x) \end{bmatrix} \quad (3)$$

is nonsingular for all  $x$ .

- (5) The nonlinear system represented by Eq. (1) is minimum phase.

Assumption 1 is a technical assumption that ensures that Lie algebra necessary for derivation of state feedback laws can be generated. Assumption 2 defines the structure of the uncertainty under consideration. This is a fairly common occurrence in chemical process models (Doyle et al., 1989a). Assumptions 3–5 restrict our theoretical development to I/O linearizable minimum phase systems with well-defined relative degrees. This represents a large class of systems in chemical process control as shown in Henson and Seborg (1997).

## 3. Robust controller design

Under the assumptions given in the previous section, there exists a diffeomorphism  $(\eta, z) = T(x)$ , given by

$$z^{(j)} = \begin{bmatrix} z_1^{(j)} \\ z_2^{(j)} \\ \vdots \\ z_{r_j}^{(j)} \end{bmatrix} = \begin{bmatrix} h_j \\ L_{\tilde{f}} h_j \\ \vdots \\ L_{\tilde{f}}^{r_j-1} h_j \end{bmatrix}, \quad j = 1, 2, \dots, m,$$

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \begin{bmatrix} \phi_{r+1} \\ \phi_{r+2} \\ \vdots \\ \phi_n \end{bmatrix}, \quad (4)$$

where  $\phi_{r+1}, \phi_{r+2}, \dots, \phi_n$  are chosen such that  $[dz_1^{(1)}, \dots, dz_{r_1}^{(1)}, \dots, dz_1^{(p)}, \dots, dz_{r_m}^{(m)}, d\phi_{r+1}, \dots, d\phi_n]$  are linearly independent. When  $r_j = \rho_j$  (relative degrees with respect to  $u$  and  $d$  are equal for all outputs), the above diffeomorphism transforms the system represented by (1) into the following nominal form:

$$\begin{aligned} \dot{\eta} &= s(z, \eta) + t(z, \eta)u + q(z, \eta)d, \\ \dot{z}_1^{(j)} &= z_2^{(j)} + L_{\Delta f} L_{\tilde{f}}^0 h_j(x), \\ \dot{z}_2^{(j)} &= z_3^{(j)} + L_{\Delta f} L_{\tilde{f}}^1 h_j(x), \\ &\vdots \\ \dot{z}_{r_j}^{(j)} &= L_{\tilde{f}}^{(r_j)} h_j(x) + L_{\Delta f} L_{\tilde{f}}^{r_j-1} h_j(x) \\ &\quad + L_g L_{\tilde{f}}^{r_j-1} h_j(x)u \\ &\quad + L_w L_{\tilde{f}}^{r_j-1} h_j(x)d \\ &\quad + L_{\Delta w} L_{\tilde{f}}^{r_j-1} h_j(x)d, \\ y_j &= z_1^{(j)}, \end{aligned} \quad (5)$$

for  $j = 1, 2, \dots, m$ .

When  $\rho_j < r_j$ , the above expressions contain time derivatives of  $d$  in the right-hand side of the equations for  $z^{(j)}$  and when  $\rho_j > r_j$ , the right-hand side of the equations for  $z^{(j)}$  are independent of  $d$  (Daoutidis and Kravaris, 1993).

If there is no uncertainty in the model ( $\Delta f=0$  and  $\Delta w=0$ ) and the disturbance inputs are measurable, the state feedback (Daoutidis and Kravaris, 1993):

$$u = (\mathcal{A}(x))^{-1} \left( v - \begin{bmatrix} L_{\tilde{f}}^{r_1} h_1(x) \\ \vdots \\ L_{\tilde{f}}^{r_m} h_m(x) \end{bmatrix} - \begin{bmatrix} L_w L_{\tilde{f}}^{r_1-1} h_1(x) \\ \vdots \\ L_w L_{\tilde{f}}^{r_m-1} h_m(x) \end{bmatrix} \right), \quad (6)$$

will induce the following decoupled set of linear systems (Daoutidis and Kravaris, 1993):

$$\begin{aligned} \dot{z}_1^{(j)} &= z_2^{(j)} \\ \dot{z}_2^{(j)} &= z_3^{(j)} \\ &\vdots \\ \dot{z}_{r_j}^{(j)} &= v_j \end{aligned} \quad (7)$$

for the case where  $r_j = \rho_j$ . Then, the external input  $v_j$  can be designed via pole placement to ensure the necessary stability and performance. In particular, if the desired set-point is the origin, one can choose  $v_j$  as follows:

$$v_j = -\beta_1^{(j)} z_1^{(j)} - \beta_2^{(j)} z_2^{(j)} - \dots - \beta_{r_m}^{(j)} z_{r_m}^{(j)}, \quad (8)$$

where the parameters  $\beta_{(i)}^j$  are tuning parameters usually selected arbitrarily such that the polynomials  $p(s)^{(j)} = s^{r_j} + \beta_{r_j}^{(j)} s^{r_j-1} + \dots + \beta_2^{(j)} s + \beta_1^{(j)}$  are Hurwitz. However due to the presence of input uncertainty and modelling errors, perturbations must be considered and the tuning parameters  $\beta^{(j)}$ s should be selected to yield both robust performance and stability.

If the disturbance is measured, a decoupling system (Daoutidis and Kravaris, 1993) can be designed whereby disturbance inputs do not affect the output. The disturbance inputs are then included in the control law in a static feedback/dynamic feedforward manner as shown in the nonlinear MIMO controller Eq. (6). However if none, or not all disturbances are measurable, then their inputs cannot be incorporated in the control law and disturbance decoupling is not possible. One way of dealing with this situation is to use disturbance inputs as a constant vector in the control law or to neglect its effects completely and treat it as uncertainty. In this work, the loss of decoupling is dealt with by posing a control problem whereby the disturbance dynamics are used in designing a linear robust controller on the outer loop.

For the case  $\rho_i = r_i$ , the feedback law:

$$u = [\mathcal{A}(x)]^{-1} \left[ v - \begin{bmatrix} L_{\tilde{f}}^{r_1} h_1(x) \\ \vdots \\ L_{\tilde{f}}^{r_m} h_m(x) \end{bmatrix} \right] \quad (9)$$

induces the following dynamics in the system with uncertainty

$$\begin{aligned} \dot{z}_1^{(j)} &= z_2^{(j)} + L_{\Delta f} L_{\tilde{f}}^0 h_j(x) \\ \dot{z}_2^{(j)} &= z_3^{(j)} + L_{\Delta f} L_{\tilde{f}}^1 h_j(x) \\ &\vdots \\ \dot{z}_{r_j}^{(j)} &= v_j + L_{\Delta f}^{(r_j-1)} h_j(x) + L_{w+\Delta w} L_{\tilde{f}}^{(r_j-1)} h_j d. \end{aligned} \quad (10)$$

For  $\rho_i > r_i$ , the disturbance term in Eq. (10) is dropped whereas for  $\rho_i < r_i$ , terms including derivatives of disturbance inputs up to order  $(r_i - \rho_i)$  will appear.

The system represented by Eq. (10) is a linear system that is subject to nonlinear perturbations. A formal Taylor’s series expansion around the desired operating point yields a *pseudo-linearized* subsystem whose Laplace transform can be modelled by the following uncertain multivariable linear system:

$$y(s) = (G + \Delta G)v(s) + (G_d + \Delta_d G_d)d(s), \tag{11}$$

where  $\Delta G$  and  $\Delta_d G_d$  are additive perturbation terms induced by the uncertainty terms in  $f$  and  $w$ , respectively.

**Remark 1.** The *pseudo-linearization* at the I/O stage involves a linear sub-system of lower dimension compared to standard Taylor series linearization of the original nonlinear plant. Furthermore, the subsequent  $\mu$ -synthesis controller design explicitly incorporates the perturbations in  $G$  and  $G_d$ , which reduce the effect of linearizing the disturbance model at steady state.

Eq. (11) is a linear system which can be represented in a block diagram as shown in Fig. 2. Using this MIMO feedback structure with multiplicative uncertainty, linear robust control techniques can be applied for controller design and analysis. The feedback structure in Fig. 2 can be reduced into the standard interconnection structure by ‘isolating’ the perturbation block  $\Delta = \text{diag}\{\Delta_u \Delta_d\}$  as shown in Fig. 3(a). The transfer function between  $\{w=d\}$  and  $z$  can be written as  $T_{zw} = \mathcal{F}_l(P, K)$  which is a linear fractional transformation (LFT) of the ‘generalized’ plant  $P$  with respect to the controller  $K$ :

$$\mathcal{F}_l(P, K) \triangleq P_{11} + P_{12}K(I - P_{22}K)^{-1}P_{21}, \tag{12}$$

$$\|T_{zw}(K)\|_\infty = \sup_\omega \bar{\sigma}(T(j\omega)), \tag{13}$$

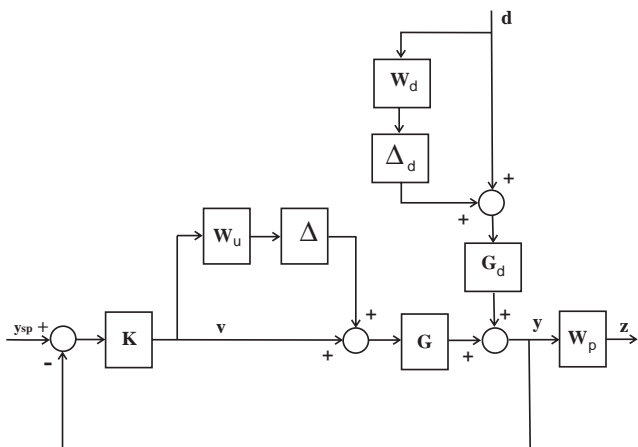


Fig. 2. Feedback structure with uncertainty.

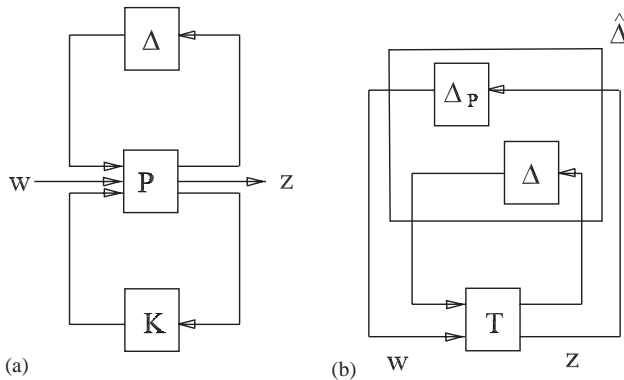


Fig. 3. Interconnection structure.

where  $\bar{\sigma}$  denotes the maximum singular value. The design problem is then reduced to *finding a controller K that meets the following objectives:* (a) *minimizes disturbance effects by minimizing the  $H_\infty$  norm between disturbance d and error z;* (b) *yields robust performance despite presence of perturbations.*

- (1) There are several techniques to solve (a) through the *model-matching* problem which includes: (i) Navelinna and Pick approach (Delsarte et al., 1981); (ii) Nehari’s problem (Glover, 1984); and (iii) State-space solution via separation principle (Doyle et al., 1989b). Usually a suboptimal controller is sought which makes  $\|T_{zw}(K)\| < \gamma$ , where  $\gamma > \mu$  is the optimal solution. The last method seems to be the most attractive from computational point of view, and its solution involves solving of two Riccati equations. Reliable algorithms for its implementation are available in the MATLAB Robust Control Tool-box.
- (2) The second part is achieved by augmenting a ‘performance’ perturbation from  $z$  to  $w$  as shown in Fig. 3(b) to form a new uncertainty block  $\hat{\Delta} = \text{diag}\{\Delta \Delta_p\}$ , and then test for robust performance using the structured singular value  $\mu$  via the following theorem.

**Theorem 1 (Skogestad and Postlethwaite, 1996).** For the uncertain system in Fig. 3(b), assume that:  $F$  is the upper LFT of  $T$  with respect to  $\hat{\Delta} = \text{diag}\{\Delta \Delta_p\}$ , i.e.,  $F_u(T, \hat{\Delta}) = N_{22} + N_{21}\hat{\Delta}(I - N_{11}\hat{\Delta})^{-1}N_{12}$ ; and that  $T$  is nominally stable then robust performance condition  $\|F_u(T, \hat{\Delta})\|_\infty < 1 \forall \|\hat{\Delta}\|_\infty \leq 1$  is equivalent to

$$\mu_{\hat{\Delta}}(N(j\omega)) < 1, \tag{14}$$

where  $\mu$  is computed with respect to the structure  $\hat{\Delta}$  and  $\Delta_p$  is a full perturbation.

The two steps above can be formulated as a  $\mu$ -synthesis problem whereby in step one, a controller  $K$  is sought that

minimizes the  $H_\infty$  norm for some fixed perturbation, and in step two, the controller is fixed and a perturbation  $\hat{\Delta}$  that minimize  $\|T_{zw}\|_\infty$  is sought. The two steps are carried out repeatedly until a satisfactory robustness margin  $\bar{\sigma}(N(j\omega)) < 1$  is obtained, or the  $H_\infty$  norm no longer decreases.

**Remark 2.** Since the zero dynamics are assumed to be stable, it is sufficient to establish that  $K$  will indeed stabilize the overall system. This can be checked by numerical simulation for some specified perturbation magnitudes or by finding a suitable Lyapunov function  $V = \xi^T \xi$  whose derivative along the trajectory of Eq. (10) satisfies  $\dot{V} \leq 0$ . Local stability results along the lines of Kolavennu et al. (2001) can be used. For rigorously proving global stability, one could use techniques presented in El-Farra and Christofides (2001).

#### 4. Application: two nonlinear CSTRs in series

A system of two non-isothermal CSTRs in series, in which the reaction  $A \rightarrow B$  occurs and temperature is manipulated by coolant flowing in reactor jackets, can be described by the standard nonlinear model like Eq. (1) as follows:

$$\dot{x} = \begin{bmatrix} -\frac{F}{V}x_1 - k_1x_1 \\ -\frac{F}{V}x_2 - \frac{\Delta H_r k_1 x_1}{\rho C_p} - \frac{UA_r}{\rho C_p V}(x_2 - x_3) \\ \frac{UA_r}{\rho_J V_J C_{P_J}}(x_2 - x_3) \\ \frac{F}{V}(x_1 - x_4) - k_2x_4 \\ \frac{F}{V}(x_2 - x_5) - \frac{\Delta H_r k_2 x_4}{\rho C_p} - \frac{UA_r}{\rho C_p V}(x_5 - x_6) \\ \frac{UA}{\rho_J V_J C_{P_J}}(x_5 - x_6) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \frac{(T_{J0_1} - x_3)}{V_J} & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{(T_{J0_2} - x_5)}{V_J} \end{bmatrix} u$$

$$+ \begin{bmatrix} \frac{F}{V} & 0 \\ 0 & \frac{F}{V} \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} d, \quad (15)$$

where  $x = [C_{A_1} \ T_1 \ T_{J_1} \ C_{A_2} \ T_2 \ T_{J_2}]^T$  is a vector of state variables;  $u = [F_{J_1} \ F_{J_2}]^T$  is manipulated variable; the disturbance  $d = [C_{A_0} \ T_{J_0}]$  is assumed to be unmeasurable. The output is chosen as temperature in the reactors  $y = [x_2 \ x_5]^T$ , and the kinetic parameters are given by  $k_i = k_0 e^{-E/RT_i}$ . The original CSTR problem (Luyben, 1990) was considered for nonlinear temperature control in a single reactor (Daoutidis and Christofides, 1995) for which model parameters are given in Table 1. This problem can be handled with available robust control techniques (e.g. Christofides and Daoutidis, 1997; El-Farra and Christofides, 2003). We can see by inspection that for disturbance:  $L_{w_1} h_1(x) = 0$  but  $L_{w_2} h_1(x) = F/V \neq 0$  and  $L_{w_1} L_f h_1(x) = F/V((- \Delta_r)k_1)/\rho C_p \neq 0$ . Hence, the relative order of the first output  $T_1$  with respect to the first disturbance  $d_1 = C_{A_0}$  and second disturbance  $d_2 = T_0$  is  $\rho_{12} = 1$  and  $\rho_{11} = 2$  respectively. For the second output we have:  $L_{w_1} h_2(x) = [0 \ 0]$  and  $L_{w_2} L_f h_2(x) = (F/V)^2 \neq 0$  and  $L_{w_1} L_f^2 h_2(x) \neq 0$ , hence  $\rho_{21} = 3$  and  $\rho_{22} = 2$ . For inputs we have  $L_g h_1(x) = [0 \ 0]$  and  $L_g h_2(x) = [0 \ 0]$  but  $L_{g_1} L_f h_1(x) = UA_r/\rho C_p V(T_{J_0} - x_3)/V_J \neq 0$  and  $L_{g_2} L_f h_2(x) = UA_r/\rho C_p V(T_{J_0} - x_5)/V_J \neq 0$ . Hence the relative order of the first output to the inputs is  $r_1 = 2$ ; and that of the second output is  $r_2 = 2$ . Choosing  $u$  as described in Section 3 leads

Table 1  
Process parameters

$C_{A_0} = 8.0$ (kg mol/m <sup>3</sup> )
$T_0 = 330.3$ (K)
$T_{J_0} = 294.4$ (K)
$F = 3.1463 \times 10^{-4}$ (m <sup>3</sup> /s)
$F_J = 3.925 \times 10^{-4}$ (m <sup>3</sup> /s)
$V = 1.3592$ (m <sup>3</sup> )
$V_J = 0.109$ (m <sup>3</sup> )
$A_r = 23.2258$ (m <sup>2</sup> )
$\rho_J = 997.95$ (kg/m <sup>3</sup> )
$E = 6.97905 \times 10^7$ (J/kg mol)
$k_0 = 1.9667 \times 10^7$ (1/s)
$R = 8331.73$ (J/(kg mol K))
$U = 851.7$ (J/(m <sup>2</sup> K))
$\Delta H_r = -6.9794 \times 10^7$ (J/kg mol)
$C_p = 3140.1$ (J/(kg K))
$C_{P_J} = 4186.8$ (J/(kg mol K))
$\rho = 800.923$ (kg/m <sup>3</sup> )

to the following nominal state feedback control law:

$$u = \begin{bmatrix} \beta_3^1 L_{g_1} L_f h_1(x) \\ \beta_3^2 L_{g_2} L_f h_2(x) \end{bmatrix}^{-1} \times \begin{bmatrix} v_1 - \sum_{j=1}^3 \beta_j^1 L_f^j h_1(x) \\ v_2 - \sum_{j=1}^3 \beta_j^2 L_f^j h_2(x) \end{bmatrix} \quad (16)$$

which induces the following dynamics:

$$\sum_{j=0}^{r_1} \beta_j^1 \frac{d^j y_1}{dt^j} = v_1 + \beta_1^1 L_w h_1(x) d + \beta_2^1 L_w L_f h_1(x) d + \beta_2^1 L_w h_1(x) d^{(1)},$$

$$\sum_{j=0}^{r_1} \beta_j^2 \frac{d^j y_2}{dt^j} = v_2 + \beta_2^2 L_w h_2(x) d. \quad (17)$$

Application of pseudo-linearization is effected by substituting the nominal steady-state values:  $C_{A_s} = 3.92 \text{ kg mol m}^{-3}$ ;  $T_{r_s} = 333 \text{ K}$ ; and  $T_{j_s} = 330 \text{ K}$  in Eq. (17). The parameters  $\beta_1^i$ ,  $\beta_2^i$  and  $\beta_3^i$  are chosen to make  $p_i(s) = \beta_3^i s^2 + \beta_2^i s + \beta_1^i$  Hurwitz and can be selected from second-order dynamics with desirable characteristics:  $\beta_3^i s^2 + \beta_2^i s + \beta_1^i = \tau_i^2 s^2 + 2\zeta_i \tau_i s + 1$ . From physical intuition the dynamics of the external plant cannot be significantly faster than the full plant. Based on open-loop dynamics of the CSTR, the parameters  $\tau = 5\text{--}7 \text{ min}$  and  $\zeta = 2\text{--}3$  seemed to yield reasonable dynamics. After pseudo-linearization and substitution of  $\beta$ 's in Eq. (17), the nominal external MIMO plant can be realized in Laplace domain as

$$y(s) = \begin{bmatrix} \frac{1}{30.25s^2 + 33s + 1} & 0 \\ 0 & \frac{1}{36s^2 + 38s + 1} \end{bmatrix} v(s) + \begin{bmatrix} \frac{0.0027}{30.25s^2 + 33s + 1} & \frac{0.42s + 0.0059}{30.25s^2 + 33s + 1} \\ 0 & \frac{1.54 \times 10^{-4}}{36s^2 + 38s + 1} \end{bmatrix} d(s). \quad (18)$$

Based on this model, the multiplicative uncertainty weights  $W_u$  in  $G$  and  $W_d$  in  $G_d$  as shown in Fig. 2 are used to design a robust controller through  $\mu$ -synthesis as described in Section 4. The generalized plant  $P$  is given by

$$P = \begin{bmatrix} 0 & 0 & 0 & W_u \\ 0 & 0 & W_d & 0 \\ W_p G & W_p G_d & W_p G_d & W_p G \\ -G & -G_d & -G_d & -G \end{bmatrix}. \quad (19)$$

The multiplicative uncertainty structure proposed by Skogestad and Postlethwaite (1996) was used to specify

uncertainty weights  $W_d$  and  $W_u$  as

$$w_i(s) = \frac{180s + 0.3}{(180/1.25)s + 1} \quad (20)$$

which means 30% relative uncertainty is allowed at steady state and at 180 s the relative uncertainty reaches 100%; with a magnitude of 1.25 at high frequency. In order to guarantee good performance and effective disturbance rejection, the  $H_\infty$  norm of the sensitivity function  $S = (I + GK)^{-1}$  is minimized especially beyond the bandwidth frequency  $\omega < \omega_B$  where disturbance dynamics are significant.

$$\|S(j\omega)W_p(j\omega)\|_\infty < 1. \quad (21)$$

Skogestad and Postlethwaite (1996) suggested the following performance weight  $W_p = ((s/M) + \omega_B/(s + \omega_B A))$  where  $M$  sets the peak of  $S$ , and  $A$  is small enough to incorporate integral action in the controller being designed. Both  $M$  and  $A$  are designed based on user knowledge of the desired shape of  $S$  in frequency domain. This approach may be valid for simple SISO cases for which closed-loop functions are well understood, however it may not be practical for complex higher order processes or MIMO cases. In this work we propose the following weight:

$$W_p(s) = \frac{\zeta(\beta s + 1)}{\beta s + \varphi}, \quad (22)$$

whose parameters are optimized using least-squares regression in frequency domain. Each  $S_i$  in the outer MIMO plant is obtained from the plant  $G_i$  and a PI controller  $K_i = k_c [1 + 1/(\tau_I s)]$  based on the external plant Eq. (18), the  $\mu$ -synthesis controller will then 'robustify' against uncertainty. Once  $K_i$  is prescribed, the sensitivity function  $S_i$  can be assumed to be known and  $\varphi$  is set small enough  $\varphi \approx 10^{-4}\text{--}10^{-7}$  to incorporate integral action. The performance weight  $W_p$  can then be written as  $W_p(s) \approx a + (b/s)$  where  $a = \zeta$  and  $b = \zeta/\beta$ . Now if the RHS of Eq. (21) is set equal to  $\{c: c < 1\}$  the inequality can be replaced by equality and at each frequency we can write:

$$\frac{c}{S(j\omega_i)} = a + \frac{b}{j\omega_i}, \quad i = 1, \dots, n_f, \quad (23)$$

where  $n_f$  is number of frequency points. The parameters  $a$  and  $b$  can be estimated by least-squares regression in frequency domain by formulating Eq. (23) in the standard form:  $\Theta = \Phi v + \varepsilon$ , where:  $\Theta = [c/(S(j\omega_1)), \dots, c/(S(j\omega_{n_f}))]^T$  is known;  $\Phi = [1 \ 1/(j\omega_i)]^T$  is a  $2 \times n_f$  matrix; and  $\varepsilon$  is estimation error vector. The vector of parameters to be estimated  $v = [a \ b]^T$  is obtained by the standard least-square solution  $v = (\Phi^T \Phi)^{-1} \Phi^T \Theta$ . Using the following settings:  $c = 0.98$ ;  $n_f = 50$ ;  $\tau_I = 180$ ;  $k_c = 0.10$ ; and  $\varphi = 5 \times 10^{-5}$  for the MIMO plant Eq. (18), the following performance weight was obtained;

$$w_p(s) = \frac{0.83(270s + 1)}{270s + 5 \times 10^{-5}}. \quad (24)$$

For all simulation experiments a sampling time of 40 s and integration step of 5 s were used. Fig. 4 compares the PI and the  $\mu$  controllers performance for a step change  $\Delta T = 5^\circ \text{C}$

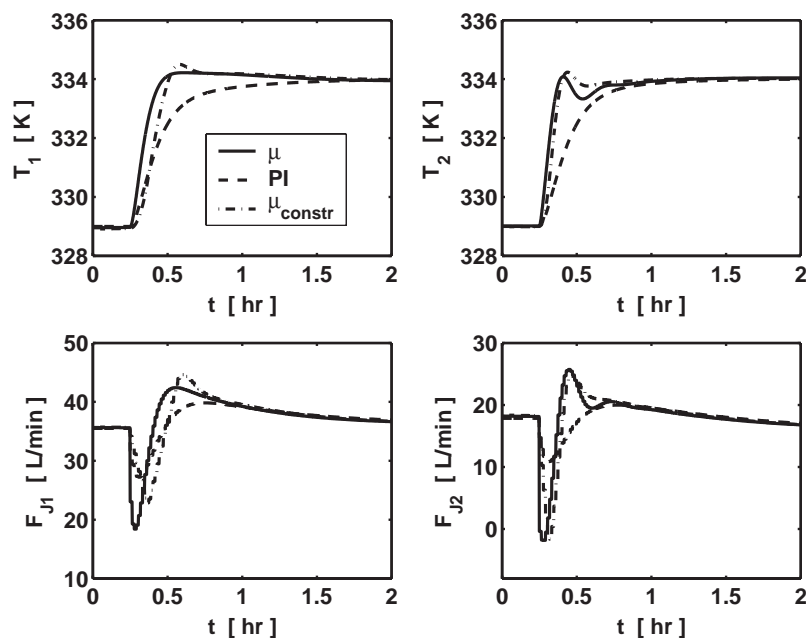


Fig. 4. Nominal response of the PI and  $\mu$ -synthesis controller (with and without input constraints) to a set-point change in reactor temperature-change.

in both reactors when the nominal process model is used. The  $\mu$  controller exhibits high controller moves at high frequencies which cause overshoots and input saturation problems for large set-points. To overcome the aggressiveness of the  $\mu$  controller, Skogestad and Postlethwaite (1996) proposed augmenting a pre-filter for extra roll-off that reduces high-frequency gains. In this work the tuning parameters  $\zeta$  and  $\tau_i$  in the external plant  $G_{ii}(s) = 1/(\tau_i^2 s^2 + 2\zeta_i \tau_i s + 1)$  were adjusted to give desired closed-loop dynamics. A value of  $\tau_i = 7$  min and a damping constant of  $\zeta_i = 2.5$  gave reasonable dynamics as shown by the solid line in Fig. 4. Since the  $\mu$  design does not penalize controller moves, performance can be further improved by imposing input constraints  $\Delta u = \text{sat}(\Delta u)$  where

$$\text{sat}(\Delta u) = \begin{cases} \Delta u_{\min} & \text{if } \Delta u \leq \Delta u_{\min}, \\ \Delta u & \text{if } \Delta u_{\min} \leq \Delta u \leq \Delta u_{\max}, \\ \Delta u_{\max} & \text{if } \Delta u \geq \Delta u_{\max}. \end{cases} \quad (25)$$

The upper and lower limits of the coolant flow rate for the constrained  $\mu$  controller plotted in Fig. 4 are  $\Delta u_{\min} = -[1.5 \ 3]^T$  and  $\Delta u_{\max} = [1.5 \ 3]^T$  liters per minute, respectively.

The  $H_\infty$  controller was designed using  $\mu$ -synthesis as described in Section 3, and the robust performance condition  $\mu(N) < 1$  is satisfied for all frequencies as shown in Fig. 5. Like the generalized plant  $P$ , the  $\mu$ -synthesis controller has 14 states, which include: four states for the second-order plants in  $G$ ; four for  $G_d$ ; and six for the first-order weights  $W_u$ ,  $W_i$ , and  $W_p$ . Using a MATLAB function for model reduction *balmr*, six states are removed and the state space controller is implemented in discrete domain. The external

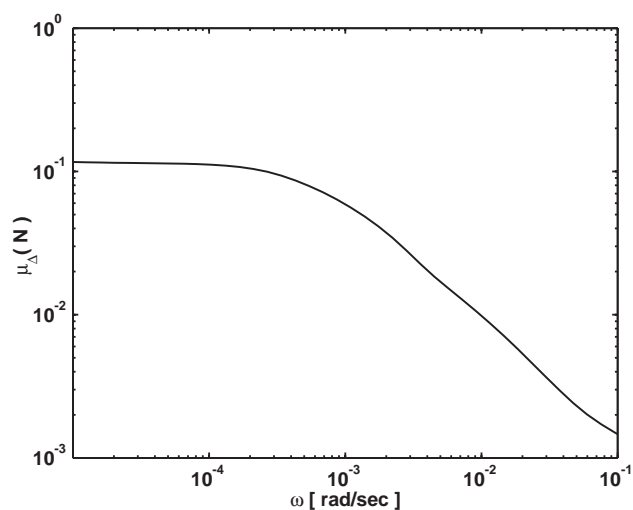


Fig. 5. Plot of  $\mu_d(N)$  versus frequency.

PI controller was designed using IMC tuning rules based on second-order dynamics of the external plant Eq. (18). The following IMC tuning rules were used: integral time constant  $\tau_I = 2\zeta_i \tau_i$ ; controller gain  $k_c = 2\zeta_i \tau_i / \lambda$ ; and a filter time constant  $\lambda = 13$  min was selected. For the sake of comparison, the same external plant parameters  $\tau$  and  $\zeta$  were used for the  $\mu$ -synthesis controller. The dynamic feedforward component- inner loop to the PI controller was implemented by approximating the derivatives of disturbance signal by a backward difference approximation in discrete domain as  $d^{(1)} = (d_k - d_{k-1}) / \Delta t$  where  $\Delta t = 40$  s is the process sampling time.

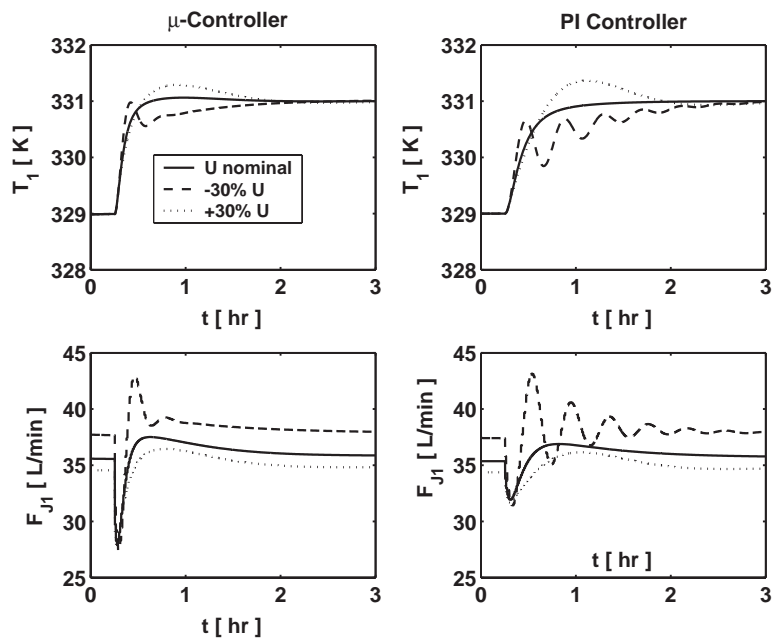


Fig. 6. Comparison of PI and  $\mu$ -synthesis controller response to a temperature set-point change in the first reactor with uncertainty in heat transfer coefficient.

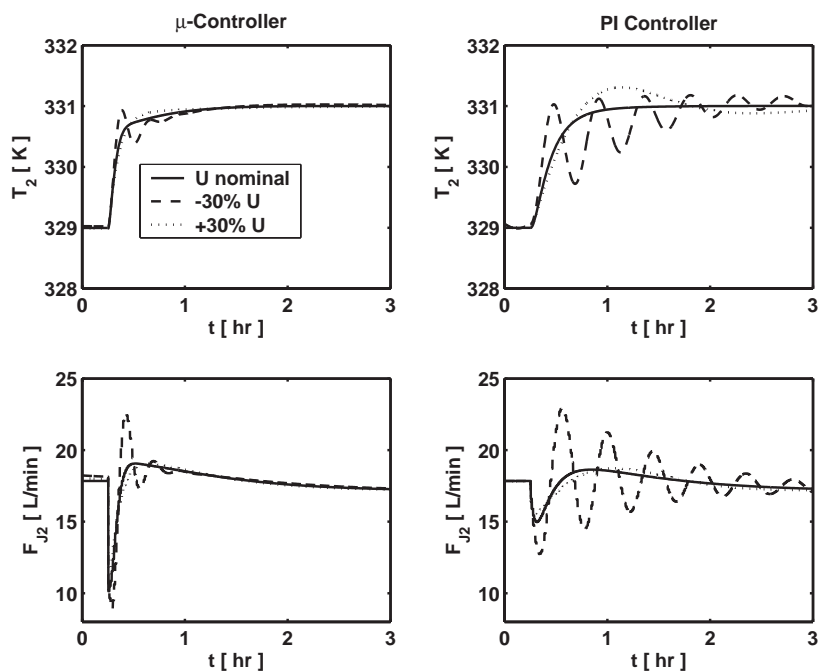


Fig. 7. Comparison of PI and  $\mu$ -synthesis controller response to a temperature set-point change in the second reactor with uncertainty in heat transfer coefficient.

Fig. 6 shows comparison of response between the PI and the  $\mu$  controller for a set-point change  $\Delta T = 2^\circ\text{C}$  in the first reactor when  $\pm 30\%$  mismatch occurs in the overall heat transfer coefficient. Fig. 7 shows similar response for the second reactor. The  $\mu$ -synthesis controller shows better overall robustness for all cases whereas the PI controller exhibits significant overshoot and under-damped behavior.

Simulation runs for parameter uncertainty in the kinetic constant  $k$  yielded similar results with the  $\mu$  parameter giving better robustness characteristics. Fig. 8 shows disturbance rejection when the inlet temperature  $d_2 = T_0$  was increased by  $10^\circ\text{C}$  and sustained for 15 min. As expected the PI controller with state feedback and dynamic feed-forward gives better overall disturbance attenuation. However the

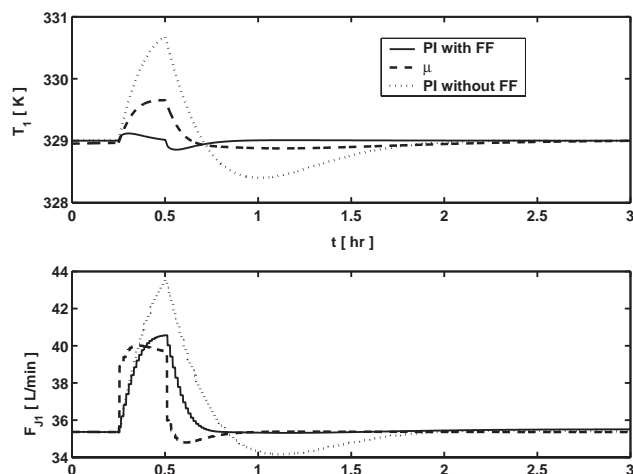


Fig. 8. Disturbance rejection by PI controller with and without feedforward compensation and  $\mu$ -synthesis controller.

$\mu$ -synthesis controller which does not use on-line disturbance information yields good performance compared to a PI controller without feed-forward. Simulation runs with changes in the inlet concentration  $d_1 = C_{A_0}$  did not have significant influence on the output variables.

## 5. Conclusions

A procedure for robust controller synthesis for MIMO nonlinear systems with uncertainty and unmeasured disturbance has been presented. The controller is implemented in a multi-loop scheme based on the I/O linearization technique, with  $\mu$ -synthesis controller in the outer loop and a nonlinear state feedback in the inner loop. The procedure was evaluated for temperature control of a non-isothermal reaction in two CSTRs in series where the inlet concentration and inlet temperature are treated as disturbances. It is shown that through proper design of performance weights and adjustable tuning parameters, the controller yields robustness in the face of parametric uncertainty as well as disturbance attenuation comparable to PI controllers using dynamic feed-forward scheme.

The controller synthesis procedure assumes full state measurement which may be a limitation during implementation. One way of addressing this is to couple open-loop observers with the state feedback design as shown in Daoutidis and Kravaris (1993) and Daoutidis and Christofides (1995). Recently, significant work in the development of closed-loop observers has been conducted by El-Farra and Christofides (2001), (2003) which could be coupled with the results developed in this paper.

## Notation

$A$  state-space matrix  
 $\mathcal{A}$  characteristic matrix

$B$  state-space matrix  
 $C$  output matrix  
 $C_A$  concentration of  $A$ , kg mol/m<sup>3</sup>  
 $C_p$  specific heat capacity, J/(kg K)  
 $d$  disturbance vector  
 $E$  activation energy, J/kg mol  
 $F$  flow rate, m<sup>3</sup>/s  
 $G$  MIMO plant  
 $\Delta H_r$  heat of reaction, J/kg mol  
 $K$  controller  
 $m$  number of output points  
 $n$  system order  
 $n_f$  number of frequency points  
 $P$  generalized plant  
 $r$  set-point  
 $R$  universal gas constant, J/(kg mol K)  
 $S$  sensitivity function  
 $t$  time, s  
 $T$  temperature, K  
 $u$  manipulated variable vector  
 $U$  heat transfer coefficient, J/(m<sup>2</sup> K)  
 $v$  external input  
 $V$  volume, m<sup>3</sup>  
 $x$  state variable vector  
 $y$  output variable vector  
 $W_p$  performance weight matrix

## Greek letters

$\beta$  tuning parameter  
 $\eta$  zero dynamic state, m K/J  
 $\zeta$  damping constant, -  
 $\Delta, \delta$  perturbation  
 $\lambda$  IMC filter constant, s  
 $\mu$  structured singular value  
 $v$  parameter vector  
 $\zeta$  transformed state vector  
 $\varphi$  tuning parameter  
 $\rho$  density, kg/m<sup>3</sup>  
 $\tau$  time constant, s  
 $\varsigma$  constant  
 $\omega$  frequency, rad/s  
 $\Gamma$  state-space matrix  
 $\bar{\sigma}$  maximum singular value

## Subscripts

0 inlet parameter  
 $c$  continuous matrix  
 $J$  jacket parameter  
 $B$  bandwidth frequency

## Acknowledgements

Support from the academic research fund (Project No. R-279-000-084-112) of the National University of Singapore is gratefully acknowledged.

## References

- Calvet, J.P., Arkun, Y., 1992. Robust stabilization of input/output linearization systems under uncertainty and disturbances. *A.I.Ch.E. Journal* 38 (8), 1145–1156.
- Christofides, P.D., Daoutidis, P., 1997. Robust control of multivariable two-time-scale non-linear systems. *Journal of Process Control* 7 (5), 313–328.
- Christofides, P.D., Teel, A.R., Daoutidis, P., 1996. Robust semi-global output tracking for nonlinear singularly perturbed systems. *International Journal of Control* 65, 639–666.
- Daoutidis, P., Christofides, P.D., 1995. Dynamic feedforward/output feedback control of nonlinear processes. *Chemical Engineering Science* 50 (12), 1889–1907.
- Daoutidis, P., Kravaris, C., 1993. Dynamic compensation of measurable disturbances in nonlinear multivariable systems. *International Journal of Control* 58, 1279.
- Delsarte, P.H., Genin, Y., Kamp, Y., 1981. The Navanlinna-Pick problem in circuit and system theory. *Journal Circuit Theory and Applications* 9, 177–187.
- Doyle, J.C., Glover, K., Khargonekar, P.P., Francis, B.A., 1989a. State-space solutions to standard  $H_2$  and  $H_\infty$  control problems. *IEEE Transactions on Automatic Control* AC-34, 831–847.
- Doyle, F.J., Packard, A.K., Morari, M., 1989b. Robust controller design for a nonlinear CSTR. *Chemical Engineering Science* 44 (9), 1929–1947.
- El-Farra, N.H., Christofides, P.D., 2001. Integrating robustness, optimality and constraints in control of nonlinear processes. *Chemical Engineering Science* 56 (5), 1841–1868.
- El-Farra, N.H., Christofides, P.D., 2003. Bounded robust control of constrained multivariable nonlinear processes. *Chemical Engineering Science* 58 (13), 3025–3047.
- Glover, K., 1984. Optimal Hankel norm approximation of linear multivariable systems and their  $L_\infty$  error bounds. *Journal Control* 39, 1115–11193.
- Henson, M.A., Seborg, D.E., 1997. *Nonlinear Process Control*. Prentice-Hall, Englewood Cliffs, NJ.
- Isidori, A., 1995. *Nonlinear Control Systems*. Springer, Berlin.
- Jiang, Z.P., Mareels, I., 1997. A small gain control method for nonlinear cascaded systems with dynamic uncertainties. *IEEE Transactions on Automatic Control* 42, 292–308.
- Kolavennu, S., Palanki, S., Cockburn, J.C., 2000. Robust control of I/O linearizable systems via multi-model  $H_2/H_\infty$  synthesis. *Chemical Engineering Science* 55 (9), 1583.
- Kolavennu, S., Palanki, S., Cockburn, J.C., 2001. Robust nonlinear control of multivariable nonlinear systems. *Chemical Engineering Science* 56 (14), 4339–4349.
- Kravaris, C., Chung, C.B., 1987. Nonlinear state feedback synthesis by global input/output linearization. *A.I.Ch.E. Journal* 33 (4), 592.
- Kravaris, C., Palanki, S., 1988. Robust nonlinear state feedback under structured uncertainty. *A.I.Ch.E. Journal* 34, 1119–1127.
- Luyben, W.L., 1990. *Process Modelling, Simulation and Control for Chemical Engineers*, 2nd Edition. McGraw-Hill, New York.
- Marino, R., Tomei, P., 1993. Robust stabilization of feedback linearizable time-varying uncertain nonlinear systems. *Automatica* 29, 181–189.
- Nijmeijer, H., van der Schaft, A.V., 1990. *Nonlinear Dynamical Control Systems*. Springer, Berlin.
- Palanki, S., Cockburn, J.C., Kolavennu, S., 2003. Robust state feedback synthesis for control of non-square multivariable nonlinear systems. *Journal of Process Control* 13 (7), 623–631.
- Skogestad, S., Postlethwaite, I., 1996. *Multivariable Feedback Control: Analysis and Design*. Wiley, West Sussex, England.