

Robust state feedback synthesis for control of non-square multivariable nonlinear systems

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Abstract

In this paper, a robust nonlinear controller is designed in the Input/Output (I/O) linearization framework, for non-square multivariable nonlinear systems that have more inputs than outputs and are subject to parametric uncertainty. A nonlinear state feedback is synthesized that approximately linearizes the system in an I/O sense by solving a convex optimization problem online. A robust controller is designed for the linear uncertain subsystem using a multi-model H_2/H_∞ synthesis approach to ensure robust stability and performance of non-square multivariable, nonlinear systems. This methodology is illustrated via simulation of a regulation problem in a continuous stirred tank reactor.

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1. Introduction

In the last two decades, there has been a significant effort in the development of the Input/Output (I/O) linearization approach to design controllers for multivariable nonlinear systems [1]. The general design approach for this class of methods has been to develop a state feedback which makes the closed-loop system *linear* in an input–output sense as well as *decoupled* (i.e., one inputs affects only one output). Then, a linear controller is designed for this linear system for desired performance. However, the state feedback design requires an exact description of the process, which is generally *not* available. Due to modeling uncertainty, input–output linearity as well as decoupling may be lost. This can result in severe degradation in controller performance as well as loss of stability. Motivated by this limitation, there has been active research in the past decade in the area of *approximate* linearization via feedback. There are different approaches to approximate linearization depending on the description of the uncertainty and the particular application under consideration. A review of various approximate linearization approaches can be

found in [2]. In this paper, we consider the development of a robust controller in the I/O setting for a system with *parametric* uncertainty. The system is approximately linearized based on nominal parameter values of the system. Then a robust outer-loop controller is designed for the “approximately linearized” system.

In the past decade, several tools have been developed in linear robust control theory [3,4]. The issue of robust controller design in the Input/Output (I/O) linearization framework for nonlinear systems has attracted attention recently. However results are available primarily for SISO systems (see [5–7] square MIMO systems (see [8–10]); the controller design issues for *non-square* multivariable systems (systems where the number of inputs is not equal to the number of outputs) in the face of parametric uncertainty are not well understood.

In this paper, we extend the multi-model approach of Kolavennu et al. [10] to non-square systems. In the multi-model H_2/H_∞ approach, the plant is described by a set of linear time invariant models and a controller is designed which achieves good quadratic performance while at the same time satisfies specified robustness criteria. This approach has been demonstrated successfully for chemical process control problems represented by linear systems [11] and square I/O linearizable nonlinear systems [10]. Non-square systems occur frequently in the chemical process industry. However, for controller

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design purposes, they are often “squared” by adding or deleting the appropriate number of inputs or outputs from the system matrix [12]. Once such systems are squared, then controller synthesis procedures developed for square systems (where the number of inputs and outputs is equal) are applied directly. However, it has been shown that there can be advantages in synthesizing a controller for the original non-square system. For instance, [13] compared square and non-square structures in a reactor application study and concluded that for their system the non-square structure was less sensitive to modeling errors due to a smaller condition number. Despite such evidence, the literature on controller design of non-square systems is sparse. Most of the available results in the literature of non-square systems are for linear systems. For instance, Treiber [14] demonstrated the application of Rosenbrock’s Direct Nyquist Array method to design a multivariable control Scheme for a ternary distillation column modeled with non-square transfer functions. A precompensator analogous to the inverse gain array for square systems was used to square and decouple the system. Lav et al. [15] utilized a similar approach for the control of a vacuum distillation column with five inputs and four outputs. Treiber and Hoffman [16] utilized a singular value decomposition (SVD) framework to synthesize a control strategy for non-square linear systems. Reeves and Arkun [12] derived a block relative gain array measure for decentralized control structure design of non-square linear systems. Chang and Yu [17] extended Bristol’s Relative Gain Array (RGA) to non-square linear systems and used this measure to assess performance based on steady state information. In the nonlinear systems area, Doyle et al. [18] investigated the design of I/O linearizing controllers for nonminimum phase nonlinear systems with two inputs and one output. The first input was used to achieve I/O linearization while the second input was used to stabilize the otherwise unstable zero dynamics. For non-square systems with more inputs than outputs and equal relative degree with respect to all inputs McLain et al. [19] derived an analytical expression, where some inputs were used for I/O linearization and the remaining inputs were used to minimize input cost.

In this paper, we propose a two step procedure that can be applied directly to non-square uncertain systems which I/O linearizable and minimum phase. The first step involves the design of a state feedback that *approximately* linearizes the system in an I/O sense by solving a convex optimization problem online. The second step involves the design of a robust controller for the uncertain linear subsystem based on a multi-objective H_2/H_∞ synthesis approach.

The remainder of the paper is organized as follows. In Section 2, the non-square multivariable robust controller design problem is formulated and the basic con-

cepts of I/O linearization of multivariable systems are reviewed. In Section 3, the effect of uncertainty on the diffeomorphism for I/O linearization is shown. The uncertain transformed system is characterized in a form that provides a framework for robust controller design. Recent results from linear robust control theory are utilized which account for parametric uncertainty. Robust stability for this controller is analyzed. In Section 4, this controller synthesis procedure is illustrated via a simulation example of a regulation problem in a multivariable continuous stirred tank reactor (CSTR). Finally, in Section 5, the major conclusions of this approach are discussed.

2. Problem formulation

Consider the following state-space model of a multi-input multi-output (MIMO) nonlinear system with parametric uncertainty

$$\begin{aligned}\dot{x} &= f(x, \theta) + g(x)u \\ y &= h(x)\end{aligned}\quad (1)$$

where $x \in \mathbb{R}^n$ is the vector of states, $u \in \mathbb{R}^m$ is the vector of manipulated inputs with scalar components u_1, u_2, \dots, u_m , $y \in \mathbb{R}^p$ the vector of measured outputs, and θ is a vector of uncertain parameters that takes values in a compact set $\Theta \subset \mathbb{R}^l$. The function f is a smooth vector function for all $\theta \in \Theta$, g consists of sufficiently smooth vector functions g_1, g_2, \dots, g_m , and h is a smooth vector function with scalar components h_1, h_2, \dots, h_p . It is assumed that $m \geq p$ since the case $p > m$ (more outputs than inputs) results in a situation where not all outputs can be controlled independently by the inputs which implies an uncontrollable situation. The cost of the inputs is represented in the following way

$$J = w_1^2 u_1^2 + w_2^2 u_2^2 + \dots + w_m^2 u_m^2 \quad (2)$$

where w_i is the cost of u_i . The objective is to design a controller utilizing all available inputs such that the total input cost at each time instant is minimized and the closed loop system is stable and certain performance objectives, e.g., tracking, disturbance rejection, etc., are satisfied for all $\theta \in \Theta$.

Most of the results available in the literature consider *square* MIMO systems. In order to describe the regulation problem precisely, we define the relative degree and the characteristic matrix specifically for *non-square* MIMO systems. These definitions for non-square systems are analogous to the definitions for square systems in Isidori [1].

Definition 1. A nonlinear MIMO system of the form (1) is said to have a relative degree r_i with respect to an output y_i if the vector

$$\begin{aligned}
 L_g L_f^k h_i(x, \theta) &\triangleq \left[L_{g_1} L_f^k h_i(x, \theta) \dots L_{g_m} L_f^k h_i(x, \theta) \right] = \mathbf{0}, \\
 k &= 0, 1 \dots r_i - 2 \\
 L_g L_f^k h_i(x, \theta) &\triangleq \left[L_{g_1} L_f^k h_i(x, \theta) \dots L_{g_m} L_f^k h_i(x, \theta) \right] \neq \mathbf{0}, \\
 k &= r_i - 1
 \end{aligned}
 \tag{3}$$

Essentially r_i is the smallest integer k for which the vector $L_g L_f^{k-1} h_i(x, \theta)$ has at least one non-zero component. This means that at least one of the inputs u_j affects the output y_i after r_i integrations.

Definition 2. If system (1) has a well defined relative degree r_i for each output y_i then the characteristic matrix of the system is the following $p \times m$ matrix,

$$\begin{aligned}
 \beta(x, \theta) &= L_g L_f^{r-1} h(x, \theta) \\
 &= \begin{bmatrix} L_{g_1} L_f^{r_1-1} h_1(x, \theta) & \dots & L_{g_m} L_f^{r_1-1} h_1(x, \theta) \\ \vdots & \ddots & \vdots \\ L_{g_1} L_f^{r_p-1} h_p(x, \theta) & \dots & L_{g_m} L_f^{r_p-1} h_p(x, \theta) \end{bmatrix}_{p \times m}
 \end{aligned}
 \tag{4}$$

If a system represented by (1) has well defined relative degree r_j for all outputs y_i with $r = \sum r_i (r_i \leq n)$ and the characteristic matrix is of full row rank, then there exists a diffeomorphism $(\eta, \xi) = T(x, \theta)$ given by

$$z^{(j)} = \begin{bmatrix} z_1^{(j)} \\ z_2^{(j)} \\ \vdots \\ z_{r_j}^{(j)} \end{bmatrix} = \begin{bmatrix} h_j(x, \theta) \\ L_f h_j(x, \theta) \\ \vdots \\ L_f^{r_j-1} h_j(x, \theta) \end{bmatrix} \quad j = 1, 2, \dots, p \tag{5}$$

$$\eta = \begin{bmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_{n-r} \end{bmatrix} = \begin{bmatrix} \phi_{r+1}(x, \theta) \\ \phi_{r+2}(x, \theta) \\ \vdots \\ \phi_n(x, \theta) \end{bmatrix} \tag{6}$$

where $\phi_{r+1}, \phi_{r+2}, \dots, \phi_n$ are chosen such that

$$\left[dz_1^{(1)}, \dots, dz_{r_1}^{(1)}, \dots, dz_1^{(p)}, \dots, dz_{r_p}^{(p)}, \dots, d\phi_{r+1}, d\phi_{r+2}, \dots, d\phi_n \right]$$

are linearly independent. This transforms the system (1) to the normal form:

$$\dot{\eta} = s(\xi, \eta, \theta) + \sum_{i=1}^m t_i(\xi, \eta, \theta) u_i \tag{7}$$

$$\left. \begin{aligned}
 \dot{z}_1^{(j)} &= z_2^{(j)} \\
 &\vdots \\
 \dot{z}_{r_j-1}^{(j)} &= z_{r_j}^{(j)} \\
 \dot{z}_{r_j}^{(j)} &= \alpha_j(\xi, \eta, \theta) + \sum_{i=1}^m \beta_{ji}(\xi, \eta, \theta) u_i
 \end{aligned} \right\} j = 1, \dots, p \tag{8}$$

where $\alpha_j = L_f^{r_j} h_j(x, \theta)$, β_{ji} is the (j,i) th entry in the characteristic matrix and $\xi = [z^{(1)}, z^{(2)}, \dots, z^{(p)}]^T$.

Eq. (8) represents p subsystems, which form the linearizable part of Eq. (1). Note that the diffeomorphism T , that puts the system in the normal form represented by Eq. (7) and Eq. (8) is a function of the parameter vector θ . If θ is uncertain, the diffeomorphism T is based on some nominal value θ_o of θ . When the nominal parameter vector θ_o is not equal to the actual plant parameter vector θ , the diffeomorphism based on θ_o does not lead to an *exactly* input–output linearizable system. This could lead to performance degradation if a conventional I/O design is used.

To overcome this loss of stability and/or performance, the dynamics of the system obtained by using a transformation based on the nominal model are studied and a robust controller design methodology for this system is derived in the next section. The following assumptions are made:

1. The nonlinear process is modeled as Eq. (1) and f and g_i are smooth vector fields, and h is a smooth scalar field.
2. The model uncertainty is assumed to be represented as uncertainty in the parameter vector θ .
3. The system represented by Eq. (1) is an input–output linearizable, minimum phase multi-variable nonlinear system, with finite relative degree r , that has a well defined normal form for all θ, \in, Θ .
4. The relative degree with respect to each output and stability properties of the original nonlinear system are not changed due to uncertainty.
5. The number of inputs is greater than or equal to the number of outputs ($m \geq p$).
6. The characteristic matrix β has full row rank.

Assumption 1 is a technical assumption that ensures that the Lie algebra required for deriving the state feedback law can be generated. Assumption 2 states that only parametric uncertainty is considered and the issue of unmodeled dynamics is not addressed in this paper. This covers a fairly large class of chemical processes where the dominant kinetics are well known but the parameter estimates are poor. Furthermore, it is assumed that the parametric uncertainty is in the vector f but not in g . This is a fairly common occurrence in chemical process models [20]. Assumptions 3 and 4 narrow down the applicability of the theoretical development to the class of systems that are input–output linearizable and minimum phase with well defined relative degree of r for all $\theta \in \Theta$. This assumption is satisfied in many practical systems with parametric uncertainty [21,22] where the input–output structure of the system is not changed with the change in model parameter values. Since most nonlinear models for chemical processes are

developed from material and energy balances, the model parameters (rate constants, heat and mass transfer coefficients etc.) are *constant*, though they may be uncertain. Consequently, in this paper, we do not consider the case where the model parameters are time-varying. The case where the parameters are time-varying can be handled by using a time-varying coordinate transformation [23] or by a gain-scheduling approach [24]. Assumption 5 states that $m \geq p$ since the case $p > m$ (more outputs than inputs) results in a situation where not all outputs can be controlled independently by the inputs which implies an uncontrollable situation. Assumption 6 ensures that the system is state controllable and a decoupling nonlinear control law exists. The case where the characteristic matrix is singular can be handled by choosing alternative outputs as shown in Soroush and Kravaris [25]. Since the characteristic matrix is a function of the outputs, [25] recommend that new outputs be chosen in such a way that the characteristic matrix is not structurally singular. The assumption of finite relative degree and nonsingular characteristic matrix ensures that the input/output linearizing law results in decoupling in the absence of parametric uncertainty [1].

3. Robust controller design

In this section a methodology for the robust controller design is outlined. First it is shown how the system (1) is transformed by diffeomorphism based on nominal parameters using Theorem 1. Then, a nominal state feedback that leads to an *approximately* linear input-output system is designed for the non-square system in Theorem 2. The uncertain transformed system is characterized in a convenient, approximate linear form using Theorem 3. It is shown via Theorem 4 that this characterization provides a framework for robust controller design using multi-objective H_2/H_∞ synthesis. Finally, it is shown via Theorem 5 that this controller stabilizes the original nonlinear system.

Theorem 1. *The system (1) with an additive model for uncertainty, i.e. of the form,*

$$f(x, \theta) = f_o(x) + \delta f(x, \theta) \quad (9)$$

under the nominal transformation $(\eta, \xi) = T(x, \theta_o)$ renders the approximately input-output linear sub-system:

$$\left. \begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + L_{\delta f} L_{f_o}^{i-1} h_j(x, \theta) \\ \dot{z}_{r_j}^{(j)} &= L_{f_o}^{r_j} + L_{\delta f} L_{f_o}^{r_j-1} h_j(x, \theta) + \sum_{i=1}^m \beta_{ji} u_i \\ y_j &= z_1^{(j)} \end{aligned} \right\} j = 1, 2, \dots, p \quad (10)$$

where f_o is the vector function f evaluated at θ_o and

$\beta_{ji}(x, \theta_o)$ is the $(j, i)^{th}$ element of the characteristic matrix evaluated at θ_o and $\xi = [z^{(1)}, z^{(1)}, \dots, z^{(p)}]^T$ where $z^{(j)} = [L_{f_o}^0 h_j, L_{f_o}^1 h_j, \dots, L_{f_o}^{r_j} h_j]^T$.

Proof. For the system (1) with additive model for uncertainty, the nominal transformation $(\eta, \xi) = T(x, \theta_o)$ is given by

$$\eta_i = \phi_i(x), 1 \leq i \leq n - r \quad (11)$$

$$z_i^{(j)} = L_{f_o}^{i-1} h_j(x), 1 \leq i \leq r_j \quad (12)$$

$$j = 1, 2, \dots, p$$

Differentiating η_i and $z_i^{(j)}$ with respect to time leads to the following equations:

$$\dot{\eta} = s(\xi, \eta, \theta) + \sum_{i=1}^m t_i(\xi, \eta, \theta) u_i \quad (13)$$

$$\dot{z}_i^{(j)} = \frac{\partial}{\partial x} [L_{f_o}^{i-1} h_j(x, \theta_o)] [f_o(x) + \delta f(x, \theta) + g(x)u] \quad (14)$$

$$1 \leq i \leq r_j \quad j = 1, 2, \dots, p$$

The outputs y_j in the new coordinate system are given by:

$$y_j = z_1^{(j)} \quad j = 1, 2, \dots, p \quad (15)$$

By the definition of relative degree [Eq. (3)]

$$L_g L_{f_o}^{i-1} h_j = 0 \quad 1 \leq i \leq r_j - 1 \quad (16)$$

$$j = 1, 2, \dots, p$$

This implies that Eq. (14) can be written as:

$$\left. \begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + L_{\delta f} L_{f_o}^{i-1} h_j(x, \theta) & 1 \leq i \leq r_j \\ \dot{z}_{r_j}^{(j)} &= L_{f_o}^{r_j} + L_{\delta f} L_{f_o}^{r_j-1} h_j(x, \theta) + \sum_{i=1}^m \beta_{ji} u_i \\ y_j &= z_1^{(j)} \end{aligned} \right\} \quad (17)$$

$$j = 1, 2, \dots, p$$

This is the same as (10). \square

Note that in Eq. (10), the m inputs u_i affect the p outputs y_j via the p equations:

$$\frac{d}{dt} \begin{bmatrix} \dot{z}_{r_1}^{(1)} \\ \dot{z}_{r_2}^{(2)} \\ \vdots \\ \dot{z}_{r_p}^{(p)} \end{bmatrix} = \begin{bmatrix} L_{f_o}^{r_1} h_1(x) + L_{\delta f} L_{f_o}^{r_1-1} h_1(x, \theta) \\ L_{f_o}^{r_2} h_2(x) + L_{\delta f} L_{f_o}^{r_2-1} h_2(x, \theta) \\ \vdots \\ L_{f_o}^{r_p} h_p(x) + L_{\delta f} L_{f_o}^{r_p-1} h_p(x, \theta) \end{bmatrix} + \begin{bmatrix} L_g L_{f_o}^{r_1-1} h_1 \\ L_g L_{f_o}^{r_2-1} h_2 \\ \vdots \\ L_g L_{f_o}^{r_p-1} h_p \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix} \quad (18)$$

If there is no parametric uncertainty, then $\delta f = 0$ and one could design a state feedback which linearizes and decouples the p equations represented by Eq. 18. This state feedback is given by the following theorem:

Theorem 2 [26]. Consider a nonsquare system of the form of Eqs. (7) and (8) with no parametric uncertainty ($\theta_o = \theta$ and $\delta f = 0$). Then, the state feedback that linearizes and decouples the p equations represented by Eq. (8) is given by

$$\begin{bmatrix} u_1 \\ u_1 \\ \vdots \\ u_m \end{bmatrix} = \beta_o^\dagger(x) \begin{bmatrix} v_1 - L_{f_o}^{r_1} h_1(x) \\ v_2 - L_{f_o}^{r_2} h_2(x) \\ \vdots \\ v_p - L_{f_o}^{r_p} h_p(x) \end{bmatrix} \tag{19}$$

where $\beta_o^\dagger(x)$ is the weighted right pseudo inverse of the characteristic matrix evaluated at θ_o .

Remark 1. The calculation of the weighted right inverse, $\beta^\dagger(x)$ is equivalent to solving a point-wise convex optimization problem that minimizes the input cost represented by Eq. (2) as shown in Kolavennu et al. [12]. Essentially the m inputs represented by u are being utilized in an “optimal blend” to generate p new inputs represented by v

We characterize the uncertainty in a suitable manner to design a outer loop controller. Input–output linearization uses coordinate transformation and state feedback to reduce the nonlinear system to a linear one. However, in the presence of uncertainties, this method does not give a perfectly linear model. Perturbations appear in the canonical form, as nonlinear functions of z , due to the presence of uncertainties. A Jacobi linearization of these nonlinear perturbations around the steady states is used so that linear robust control techniques can be used. It may be noted that this is different from the Jacobi linearization of the original nonlinear system. Only the perturbations arising due to uncertainties are linearized but not the whole model.

Theorem 3. The uncertain system of the form of Eq. (10) under the nominal state feedback (19) can be characterized as

$$\begin{aligned} \dot{\xi} &= A(\theta)\xi + \sum_{i=1}^m b_i v_i + W_d d \\ y &= C\xi \end{aligned} \tag{20}$$

where $\|d_i\|_2 \leq 1$ are the non-linear perturbations represented as external bounded disturbances and W_d is a linear time invariant stable, minimum-phase weight.

Proof. From Theorem 2, application of the nominal state feedback (19) to Eq. (10) results in the following:

$$\left. \begin{aligned} \dot{z}_i^{(j)} &= z_{i+1}^{(j)} + L_{\delta f} L_{f_o}^{i-1} h_j(x, \theta) \quad 1 \leq i \leq r_j - 1 \\ \dot{z}_{r_j}^{(j)} &= L_{\delta f} L_{f_o}^{r_j-1} h_j(x, \theta) + v_j \end{aligned} \right\} j = 1, 2, \dots, p \tag{21}$$

By a formal Taylor series expansion we can write

$$L_{\delta f} L_{f_o}^{i-1} h_j(x, \theta) = \delta_i^{(j)}(\theta) z^{(j)} + \tilde{\delta}_i^{(j)}(\eta, z^{(j)}, \theta) \quad \begin{aligned} 1 \leq i \leq r_j \\ j = 1, 2, \dots, p \end{aligned} \tag{22}$$

where $\delta_i^{(j)}(\theta) z^{(j)}$ are the first order terms in the Taylor series expansion and $\tilde{\delta}_i^{(j)}(\eta, z^{(j)}, \theta)$ contain the remaining terms. Substituting Eq (22) in Eq. (21), we get:

$$\frac{d}{dt} \begin{bmatrix} z_1^{(j)} \\ z_2^{(j)} \\ \vdots \\ z_{r_j}^{(j)} \end{bmatrix} = \underbrace{\begin{bmatrix} z_2^{(j)} + \delta_1^{(j)}(\theta) z^{(j)} \\ z_3^{(j)} + \delta_2^{(j)}(\theta) z^{(j)} \\ \vdots \\ z_{r_j}^{(j)} + \delta_{r_j-1}^{(j)}(\theta) z^{(j)} \\ \delta_{r_j}^{(j)}(\theta) z^{(j)} \end{bmatrix}}_{\text{linear in } z^{(j)}} + \underbrace{\begin{bmatrix} \tilde{\delta}_1^{(j)} \\ \tilde{\delta}_2^{(j)} \\ \vdots \\ \tilde{\delta}_{r_j-1}^{(j)} \\ \tilde{\delta}_{r_j}^{(j)} \end{bmatrix}}_{\text{nonlinear in } z^{(j)}} + \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ \vdots \end{bmatrix} \tag{23}$$

$j = 1, 2, \dots, p$

Define the following vectors:

$$\xi = \begin{bmatrix} z_1^{(1)}, z_2^{(1)}, \dots, z_{r_1}^{(1)}, z_1^{(2)}, z_2^{(2)}, \dots, z_{r_2}^{(2)}, \dots, z_1^{(p)}, z_2^{(p)}, \dots, z_{r_p}^{(p)} \end{bmatrix}^T \tag{24}$$

$$b_i = e_k \quad \text{where } k = \sum_{j=1}^i r_j \tag{25}$$

and e_k is the k th basis vector in \mathbb{R}^r .

$$\Delta_A = \begin{bmatrix} \tilde{\delta}_1^{(1)}, \tilde{\delta}_2^{(1)}, \dots, \tilde{\delta}_{r_1}^{(1)}, \tilde{\delta}_1^{(2)}, \tilde{\delta}_2^{(2)}, \dots, \tilde{\delta}_{r_2}^{(2)}, \dots, \tilde{\delta}_1^{(p)}, \tilde{\delta}_2^{(p)}, \dots, \tilde{\delta}_{r_p}^{(p)} \end{bmatrix} \tag{26}$$

$$C = \begin{bmatrix} C_1 & 0 & \dots & 0 \\ 0 & C_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & C_m \end{bmatrix} \tag{27}$$

C_i is a row vector of length r_i whose first element is 1 and the rest all are zeros.

Substituting the above vectors in Eq. (23) results in the following:

$$\begin{aligned} \dot{\xi} &= A(\theta)\xi + \Delta_A + \sum_{i=1}^p b_i v_i \\ y &= C\xi \end{aligned} \tag{28}$$

where the matrix $A(\theta)$ is obtained by factoring out the vector ξ from the linear terms in Eq. (23).

The non-linear perturbations Δ_A can be represented as external bounded disturbances. Let $d_i \in L_2[0, \infty)$, such that $\|d_i\|_2 \leq 1, 1 \leq i \leq r$. Stable linear time invariant weights W_{d_i} are chosen such that

$$\|\tilde{\delta}_i\|_2 \leq \|W_{d_i}d_i\|_2, \quad 1 \leq i \leq r \tag{29}$$

Then the effects of Δ_A can be represented by $W_d d$, where $W_d = \text{diag}(W_{d_1}, \dots, W_{d_r})$. The uncertainty in the input Δ_β is a function of the vector of uncertain parameters. This reduces (28) to (20). \square

Remark 2. We assume that the nonlinear perturbations Δ_A are bounded. This adds conservativeness to the controller design as it may not be possible to find tight bounds for a given system.

To complete the design we must find a robustly stabilizing controller for the uncertain system (20). This is a linear robust control problem that can be solved via multi-objective optimization techniques such as mixed H_2/H_∞ synthesis with pole placement constraints. This technique can be used for robust design when the linear fractional representation of the plant is affine in θ . The multi-model H_2/H_∞ state-feedback synthesis places the poles such that the system has good performance for all values of θ . This problem is represented in Fig. 1. The term, w , contains all external disturbances, e.g. d , and Z_2 and Z_∞ contain the relevant errors signals that we want to maintain small with respect to the two-norm (average) and ∞ -norm (worst case), respectively. The generalized plant $G(\theta)$ represents the plant model together with performance and normalization weights. The objective is to find a stabilizing controller K such that

$$a\|T_{Z_\infty w}\|_\infty + b\|T_{Z_2 w}\|_2 \tag{30}$$

is minimized, for all $\theta \in \Theta$, where $T_{Z_\infty w}$ and $T_{Z_2 w}$ are linear operators mapping w to Z_∞ and w to Z_2 respectively and a, b are positive numbers representing the trade-off between the H_2/H_∞ objectives.

For the problem to be tractable, G should be affine in θ . If the matrix G is not affine, it poses a non-convex, infinite dimensional optimization problem. For this reason, the uncertain state space model (20) is represented as a polytopic family of systems where the state

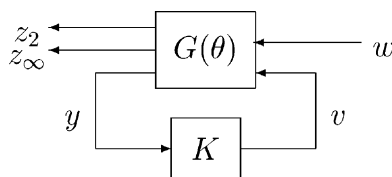


Fig. 1. Multi-model H_2/H_∞ synthesis problem.

space matrices are affine functions of the uncertain parameters i.e. of the form

$$A(\theta) = A_0 + \theta_1 A_1 + \dots + \theta_k A_k + \dots + \theta_q A_q \tag{31}$$

where q is number of uncertain parameters. Then the multi-objective problem (30) is solved by Linear Matrix Inequalities (LMI) using the following theorem.

Theorem 4 [4]. Given a polytopic family of LTI systems, of the form

$$\dot{\xi} = A\xi + B_1 d + \sum_{i=1}^p b_i v_i \tag{32}$$

$$Z_\infty = C_1 \xi + D_{11} d + D_{12} v \tag{33}$$

$$Z_2 = C_2 \xi + D_{22} v \tag{34}$$

The state feedback $v = K\xi$ that robustly stabilizes the above system and minimizes the performance objective is given by $K = YX^{-1}$, where X and Y are obtained by solving the following LMI formulation of the multi-objective state feedback synthesis problem:

Minimize $\alpha\gamma^2 + b \text{Trace}(Q)$ for all k over Y, X, Q and γ^2 satisfying

$$\begin{pmatrix} \tilde{A}(X, Y) & B_{1k} & XC_1^T + Y^T D_{12}^T \\ B_{1k}^T & -I & D_{11}^T \\ C_1 X + D_{12} Y & D_{11} & -\gamma^2 I \end{pmatrix} < 0 \tag{35}$$

$$\begin{pmatrix} Q & C_2 X + D_{22} Y \\ XC_2^T + Y^T D_{22}^T & X \end{pmatrix} > 0 \tag{36}$$

$$\text{Trace}(Q) < v_0^2 \tag{37}$$

$$\gamma < \gamma_0^2 \tag{38}$$

$$f_D(X, Y) < 0 \tag{39}$$

where $\tilde{A}(Z, Y) = A_k X + X A_k^T + B_k Y + Y^T B_k^T, B = [b_1 \dots b_2 \dots \dots b_m], A_k, B_{1k}, B_k$ are coefficients in the polytopic representation [as shown in Eq. (31)] of the parameter dependent state matrices A, B_1 , and B respectively, and γ_0 and v_0 are upper bounds on the H_∞ and H_2 norms respectively and $f_D(X, Y)$ specifies the pole placement constraints.

The minimization problem posed by Theorem 4 can be solved using the standard software for semi-definite optimization (e.g. LMI control toolbox in MATLAB [3]).

If a linear controller K cannot be found by solving the optimization problem (30) in Theorem 4, this does not imply that a robustly stabilizing controller does not exist for the original uncertain nonlinear system. This situation can arise when a bound on $\tilde{\delta}$ cannot be established or when the bound on $\tilde{\delta}$ is so large that the performance level γ cannot be satisfied for the uncertainty.

3.1. Controller design procedure

The procedure for designing a robust controller for the uncertain nonsquare multivariable system represented by Eq. (1) is as follows.

1. Calculate the relative degree r_i for each of the p outputs.
2. Transform the original system to the normal form (13) and (10) using a diffeomorphism based on *nominal* parameters.
3. Introduce p new reference inputs to approximately cancel the nonlinearities in subsystem (10).
4. Use the linear multivariable control technique presented in Theorem 4 to design a controller for the resulting linear subsystem (20) in terms of the external reference inputs v_i .
5. At each time step, solve a convex optimization problem numerically to compute the weighted right pseudo inverse β_o^* .
6. Use the optimal input solution in the approximately linearizing inner-loop controller.

Note that in this procedure, the external linear loop is designed *first* and the I/O linearizing inner loop is designed *later* unlike the case for square multivariable systems where the inner loop is designed first.

Now, it is shown that the feedback controller found by multi-objective synthesis robustly stabilizes the original nonlinear system.

Theorem 5. Consider a system of the form of Eq. (20) and assume that

1. θ is in a compact set
2. $\|A_v\|_\infty < 1$
3. $\|d_i\|_2 \leq 1$
4. W_d LTI stable, minimum-phase weight.

If a controller K robustly stabilizes the system represented by (20), then this controller also robustly stabilizes the nonlinear system represented by (1).

The proof is along the lines of the stability proof in Kolavennu et al. [10] and is omitted here for the sake of brevity.

Remark 3. The stability result obtained from Theorem 5 is valid only locally for sufficiently small initial conditions.

4. Illustrative example

Consider the following process model of a reversible reaction $A \rightleftharpoons B$ taking place in a constant volume CSTR [19]:

$$\begin{aligned} \frac{dC_A}{dt} &= \frac{F}{V}(C_{Ai} - C_A) - k_1(T)C_A + k_2(T)C_B \\ \frac{dT}{dt} &= \frac{F}{V}(T_i - T) + \frac{(-\Delta H)}{\rho C_p}(k_1(T)C_A + k_2(T)C_B) \end{aligned} \quad (40)$$

where $k_i(T) = A_i \exp(-\frac{E_i}{RT})$.

The objective is to control the concentration C_B to a value of 0.4 by manipulating the feed temperature T_i and feed concentration C_{Ai} . Thus

$$y = C_B \quad (41)$$

The uncertain parameter is A_2 , it has a nominal value of 5.0×10^3 and can vary between 3.0×10^3 and 7.0×10^3 . The relative cost of both inputs is assumed to be the same and the robust controller has to use a blend of these inputs in an optimal way to regulate the output.

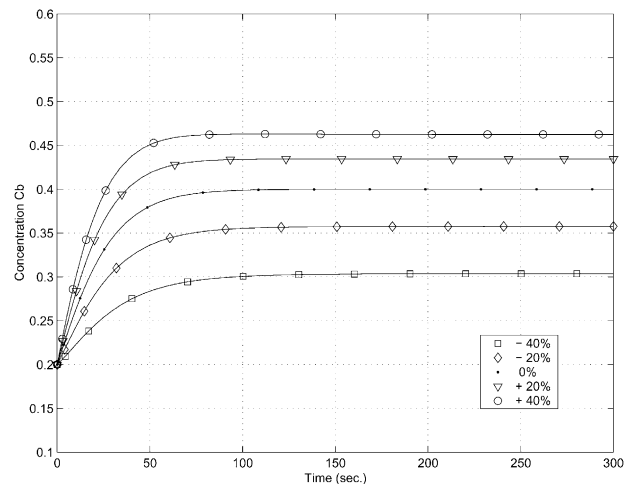


Fig. 2. Output profile for conventional I/O controller.

Table 1
Variables used in illustrative example

Symbol	Value
$-\Delta H_1$	4.5e5 KJ
$-\Delta H_2$	5.0e5 KJ
$-\Delta H_3$	6.0e5 KJ
A_1	2.0e6 min ⁻¹
A_2	1.2e6 min ⁻¹
A_3	1.2e6 Kmole ⁻¹ min ⁻¹
E_1	5.0e4 KJ
E_2	6.5e4 KJ
E_3	5.7e4 KJ
ρ	1000 Kg/m ³
C_p	4.2 KJ/Kg K
V	0.01 m ³
F/V	0.1 min ⁻¹
T_0	295 K
C_{A_0}	1 Kmole/min

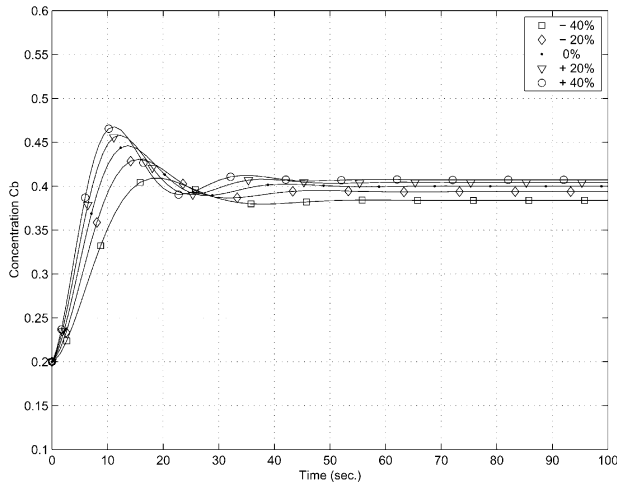


Fig. 3. Output profile for Robust controller.

Using Theorems 1–3 and the numerical values given in Table 1 results in the following linear subsystem

$$\frac{d}{dt} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} 0.00043\theta & 1 + 0.002\theta \\ -0.000141\theta & -0.006531\theta \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} v + \begin{bmatrix} 0 \\ 1 \end{bmatrix} d \quad (42)$$

where $z_1 = C_B - C_{B_s}$ and $z_2 = -\left(\frac{F}{V} + k_2\right)C_B + k_1C_A$ and C_{B_s} is the steady state value of species B . The robust controller described in the previous section is designed for this example. The H_∞ objective is to minimize the influence of the disturbance on the output and the H_2 objective is to minimize the effect of the disturbance on the vector $[z_1 \ z_2 \ v]$. Using the LMI control toolbox from MATLAB the following control law is obtained

$$v = -0.046z_1 - 0.29z_2 \quad (43)$$

A conventional I/O linearizing controller for this system was developed by McLain et al. [19] for the case where there is no uncertainty and is given by

$$v = -0.0044z_1 - 0.133z_2 \quad (44)$$

The performance of the nominal and the robust controllers are shown in Figs. 2 and 3. The parameter A_2 is varied by $\pm 40\%$ in the process model. It is seen that the conventional I/O controller performance degrades significantly in the presence of parametric uncertainty while the robust controller performs well in the face of uncertainty.

5. Conclusions

A state feedback synthesis procedure was developed for a class of uncertain MIMO nonlinear systems based on I/O linearization and multi-objective H_2/H_∞ synthesis. The procedure is applicable for minimum phase

systems that are I/O linearizable. The inner loop is based on nominal parameters of the model. The outer loop is designed to provide robustness to uncertainties in plant parameters. The controllers can be designed using off-the-shelf software and do not require restrictive matching conditions to be satisfied. This methodology was illustrated via simulation of a regulation problem in a CSTR.

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