# QUASI-ISOMETRIES AND AMALGAMATIONS OF TAME COMBABLE GROUPS

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Abstract. We study the property of tame combability for groups. We show that quasi-isometries preserve this property. We prove that an amalgamation, $A *_C B$, where $C$ is finitely generated, is tame combable iff both $A$ and $B$ are. An analogous result is obtained for HNN extensions. And we show that all one-relator groups are tame combable.

## 1. Introduction

Casson has recently come up with a condition, $C_2$, on finite presentations of groups for which the following theorem holds (see [3]):

**Theorem(Casson).** *Suppose $M$ is a closed $\mathbb{P}^2$-irreducible three-manifold with infinite fundamental group. If $\pi_1(M)$ has a finite presentation that satisfies condition $C_2$ then the universal cover of $M$ is $\mathbb{R}^3$.*

The condition $C_2$ is not necessarily preserved by change of presentation. In [2] we develop the qsf property which is invariant under change of presentation. Hence it is natural to speak of the class of qsf groups.

In [5], Mihalik and Tschantz define the class of tame combable groups. They show that this class contains all asynchronously automatic and semi-hyperbolic groups. Further they prove that tame combable groups are qsf. It is a result of Tschantz's (unpublished) that not all qsf groups are tame combable.

In this paper we show that tame combability is a geometric property, in the sense that it is preserved by quasi-isometries. We also prove that an amalgamation, $A *_C B$, where $C$ is finitely generated, is tame combable iff both $A$ and $B$ are. And we show that an HNN extension, $A *_C \phi$, with $C$ finitely generated, is tame combable iff $A$ is. Further we are able to sharpen a result in [2] and show that all one-relator groups are tame combable.

As in [2], we work with CW-complexes whose two-cell attaching maps are PL. Maps between complexes are assumed to be cellular. Given a subcomplex $A$ of a complex $Z$, the $n$-neighborhood is defined to be the iterated star without subdivision, $\mathrm{Star}^n(A)$.

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## 2. Preliminaries

In [5], two definitions of tame combability are given and shown to be equivalent. One makes explicit use of combings; the second appeals to a concept developed by Tucker in his work on the missing boundary problem for 3-manifolds (see [6]). We find it convenient to work with this second formulation. To distinguish it from the first, we will refer to it as the Tucker property.

Let $X$ be a finite complex and $\tilde{X}$ be its universal cover. We will say that $X$ has *the Tucker property* or that $X$ *is Tucker* iff, given any finite subcomplex $C \subset \tilde{X}$, $\pi_1(\tilde{X} \setminus C)$ is finitely generated for any choice of basepoint (i.e., for each component of $\tilde{X} \setminus C$). It is a consequence of theorems 1 and 2 in [5] that this only depends on $\pi_1(X)$, i.e., if $X_1$ and $X_2$ are finite complexes with $\pi_1(X_1) = \pi_1(X_2)$ then $X_1$ has the Tucker property iff $X_2$ has the Tucker property. So we may view the Tucker property as a property of finitely presented groups, and will say that $G$ is Tucker iff some (and hence any) finite complex $X$, with $\pi_1(X) = G$, has the Tucker property.

We will find it convenient to apply the above definition to each end of $\tilde{X}$ separately. Recall that an end $e$ of a space $Z$ is a choice, for each compact $C \subset Z$, of a non-compact component, $e(C)$, of $Z \setminus C$, such that $C_1 \subset C_2$ implies $e(C_2) \subset e(C_1)$. Write $\mathcal{E}(Z)$ for the set of ends. A proper map $f : Z \to W$ induces a map on the set of ends $f_{\mathcal{E}} : \mathcal{E}(Z) \to \mathcal{E}(W)$ as follows: Let $e$ be an end of $Z$ and $C \subset W$ compact. Since $f$ is proper, $f^{-1}(C)$ is a compact subset of $Z$. Write $U$ for the component $e(f^{-1}(C))$ of $Z \setminus f^{-1}(C)$. Then $f_{\mathcal{E}}(e)(C)$ is defined to be the component of $W \setminus C$ containing the connected set $f(U)$. Clearly, this makes $\mathcal{E}$ into a functor.

We will say that an end $e$ has *the Tucker property* iff, given a finite subcomplex $C \subset \tilde{X}$, $\pi_1(e(C))$ is finitely generated. (We will also say that $e$ *is Tucker*.) Note that $X$ has the Tucker property iff each end of $\tilde{X}$ does.

Recall that metric spaces $A$ and $B$ are quasi-isometric if there are functions $f : A \to B$ and $g : B \to A$ and there are constants $k$ and $\epsilon$ such that

$$\begin{aligned} d(f(a_1), f(a_2)) &\leq k \cdot d(a_1, a_2) + \epsilon \\ d(g(b_1), g(b_2)) &\leq k \cdot d(b_1, b_2) + \epsilon \\ d(f \circ g(b), b) &\leq k \\ d(g \circ f(a), a) &\leq k \end{aligned}$$

If $G$ and $H$ are finitely generated groups then we say that $G$ and $H$ are quasi-isometric if, for some choice of generators, the Cayley graphs are quasi-isometric, with the metrics being the path metrics. It turns out that this does not depend on the generating sets chosen. Also a bit more is true (see [1]). If $X$ and $Y$ are finite two-complexes with quasi-isometric fundamental groups then, writing $\tilde{X}$ and $\tilde{Y}$ for the universal covers, there are proper cellular maps $\alpha : \tilde{X} \to \tilde{Y}$ and $\beta : \tilde{Y} \to \tilde{X}$ and a constant $N$ such that we have

$$d(\alpha(x_1), \alpha(x_2)) \leq N \cdot d(x_1, x_2) \tag{1}$$

$$d(\beta(y_1), \beta(y_2)) \leq N \cdot d(y_1, y_2) \tag{2}$$

$$d(\alpha \circ \beta(y), y) \leq N \tag{3}$$

for all vertices $x_1, x_2, x \in \tilde{X}$ and $y_1, y_2, y \in \tilde{Y}$, where $d$ is the appropriate path metric (see [1]). We will say that $(\alpha, \beta)$ is *an extended combinatorial quasi-isometry with constant* $N$.

## 3. Quasi-isometries

We saw in [1] that, given a extended combinatorial quasi-isometry $(\alpha, \beta)$, the induced maps on the sets of ends, $\alpha_{\mathcal{E}}$ and $\beta_{\mathcal{E}}$, are inverses to each other and map simply connected or semistable ends to ends of the same type. We will now prove an analogous result for the Tucker property.

**Theorem 3.1.** *Suppose that $(\alpha, \beta)$ is an extended combinatorial quasi-isometry between $\tilde{X}$ and $\tilde{Y}$, where, as above, $\tilde{X}$ and $\tilde{Y}$ are the universal covers of finite complexes. Then $\alpha_{\mathcal{E}}$ and $\beta_{\mathcal{E}}$ map Tucker ends to Tucker ends.*

*Proof.* Suppose $e$ is an end of $\tilde{X}$ having the Tucker property. We will show that $\alpha_{\mathcal{E}}(e)$ also has the Tucker property. By symmetry, this will suffice.

Let $N$ be the constant associated to $(\alpha, \beta)$. Take $C$ to be a finite subcomplex of $\tilde{Y}$ and $K$ to be a finite subcomplex of $\tilde{X}$ containing the compact subset $\alpha^{-1}(C)$. Write $U = e(K)$, a component of $\tilde{X} \setminus K$, and write $V$ for the component of $\tilde{Y} \setminus C$ containing $\alpha(e(\alpha^{-1}(C)))$. This is the same component as that containing $\alpha(U)$. Thus $V = \alpha_{\mathcal{E}}(e)(C)$. We need to see that $\pi_1(V)$ is finitely generated.

By [1, lemma 3.1], there is a constant $m'$ such that any edge loop $w$ in $\tilde{Y}$ of length $\leq m = N^2 + 2N + 1$ is null-homotopic in the $m'$-neighborhood of $w$. Hence any edge loop lying entirely outside of the $m'$-neighborhood of $C$, of length $\leq N^2 + 2N + 1$, is null-homotopic outside of $C$.

Let $C_1$ be the $(N + m')$-neighborhood of $C \cup \alpha(K)$. Write $V_1$ for a component of $\tilde{Y} \setminus C_1$ contained in $V$. There are finitely many such components. Further, $V$ may be gotten from the union of such components by adding finitely many cells. Hence it is sufficient to show for all such $V_1$, that the subgroup of $\pi_1(V)$ generated by edge loops contained in $V_1$ (i.e., the image of $\pi_1(V_1)$) is finitely generated.

A simple argument using formula (4) shows that we can find a vertex $y_0 \in V_1$ with $\beta(y_0) \in U$. Formula (3) then implies that $\alpha(\beta(y_0)) \in V$. Further, $\alpha(\beta(y_0))$ lies outside of the $m'$-neighborhood of $C$.

Choose an edge-path $p$ in $V$ of length $\leq N$ from $y_0$ to $\alpha(\beta(y_0))$. By hypothesis, $\pi_1(U, \beta(y_0))$ is finitely generated. Let $\rho_1, \ldots, \rho_n$ be a generating set of edge loops. Consider, for $i = 1, \ldots, n$, the edge loops $\tau_i = p \cdot \alpha(\rho_i) \cdot p^{-1}$ in $V$. We will show that these generate the image of $\pi_1(V_1, y_0)$ in $\pi_1(V, y_0)$. As mentioned above, this will suffice.

Let $w$ be an edge loop in $V_1$ based at $y_0$. By formula (3), $\alpha(\beta(w))$ misses $\alpha(K)$. Hence $\beta(w)$ misses $K$. By connectedness, $\beta(w)$ is an edge loop in $U$, based at $\beta(y_0)$. Thus there is a basepoint preserving homotopy in $U$ from the loop $\beta(w)$ to some product of the $\rho_i$'s and their inverses. Applying the map $\alpha$ yields a base preserving homotopy in $V$ from $\alpha(\beta(w))$ to some product of the $\alpha(\rho_i)$'s and their inverses. It follows that the edge loop $p \cdot \alpha(\beta(w)) \cdot p^{-1}$, based at $y_0$, is homotopic in $V$, rel its basepoint, to a product of the $\tau_i$'s and their inverses. Now we need only see that $p \cdot \alpha(\beta(w)) \cdot p^{-1}$ is homotopic in $V$, rel its basepoint, to $w$.

Let $w = e_1 \cdots e_t$, where $e_i$ is an edge from $y_{i-1}$ to $y_i$. Note that $y_t = y_0$. By formulas (1) and (2), the edge path $\alpha(\beta(e_i))$ is of length $\leq N^2$. Write paths, $p_i$, in $V$ of length $\leq N$ from $y_i$ to $v_i$. We may take $p_0 = p$. Fix $i$. Then the loop $e_i \cdot p_i \cdot \alpha(\beta(e_i))^{-1} \cdot (p_{i-1})^{-1}$ is of length $\leq N^2 + 2N + 1$ and, by formula (3), it lies outside of the $m'$-neighborhood of $C$. Hence, by our choice of $m'$, this loop is null-homotopic outside of $C$. So there is a Van-Kampen diagram for this loop with image lying in $V$. Glueing together such diagrams shows that $w$ is homotopic in $V$, rel its basepoint, to $\alpha(\beta(w))$. $\square$

The preceding can be used to show that the Tucker property for finitely presented groups does not depend on the finite complex chosen. Also, we immediately get:

**Corollary 3.2.** *Suppose $G$ and $H$ are quasi-isometric finitely presented groups. Then $G$ is Tucker iff $H$ is Tucker.*

## 4. AMALGAMATIONS

Suppose $X$ is a complex and $S \subset X$ is a subcomplex. We say that $S$ is *two-sided or bicollared in $X$* if a regular neighborhood of $S$ in $X$ is homeomorphic to $S \times [-1, 1]$, with $S$ corresponding to $S \times 0$. Given a two-sided subcomplex $S$ in $X$, one calls $X_S = X \setminus (S \times (-1, 1))$ the result of cutting $X$ along $S$.

A subcomplex is said to be *$\pi_1$-injective* iff for any choice of basepoint (i.e., for any component) the inclusion map induces an injection of fundamental groups.

We now turn to our amalgamation result. Observe that if $S \subset X$ is two-sided and $\pi_1$-injective in $X$ then $X_S$ is also $\pi_1$-injective in $X$. And note that $X_S$ need not be connected. It is Tucker iff each of its components is.

**Theorem 4.1.** *Suppose $X$ is a finite complex and $S \subset X$ is a two-sided subcomplex. Assume that $X_S$ is $\pi_1$-injective in $X$. Then $X$ is Tucker iff $X_S$ is Tucker.*

*Proof.* Write $p : \tilde{X} \to X$ for the universal cover. Observe that $p^{-1}(S)$ is two-sided in $\tilde{X}$. Let $N(S) = p^{-1}(S) \times [-1, 1]$ be a regular neighborhood of $p^{-1}(S)$. We will call a subcomplex $D$ *transverse* if $D \cap N(S) = (D \cap p^{-1}(S)) \times [-1, 1]$.

We will use $(\tilde{X}_S)_i$ to mean a typical component of $p^{-1}(X_S)$.

Assume that $X_S$ is Tucker. Let $C$ be a finite subcomplex of $\tilde{X}$. Write $U$ for one of the components of $\tilde{X} \setminus C$.

We can add finitely many cells to $C$ in order to obtain a transverse subcomplex $C_1$. Let $U_1$ be some component of $\tilde{X} \setminus C_1$ contained in $U$ (there may be more than one). Since $C_1$ was gotten from $C$ by adding finitely many cells, it suffices for this half of the proof to see that $\pi_1(U_1)$ is finitely generated.

Since $C_1$ is transverse, there are two-sided subsets of $U_1$ cutting it into pieces, each of which is either of the form "some component of $(\tilde{X}_S)_i \setminus (C_1 \cap \tilde{X}_S)$" or "$(\tilde{X}_S)_i$". The pieces of the second type are simply-connected. By our hypothesis, each of the pieces of the first type has finitely generated fundamental group. If it turned out that the subsets we were cutting along were $\pi_1$-injective then it would follow that $\pi_1(U_1)$ was a tree product of the fundamental groups of the pieces. But in any case, a simple induction argument using Van Kampen's theorem implies that $\pi_1(U_1)$ is generated by the generators of the fundamental groups of the pieces. Since there are only finitely many pieces of the first type, the desired result follows.

Assume that $X$ is Tucker. We want to see that each component of $X_S$ is Tucker.

Again, we enlarge $C$ to a transverse finite subcomplex $C_1$ and take $U_1$ to be some component of $(\tilde{X}_S)_i \setminus C_1$ contained in $U$. It suffices to see that $\pi_1(U_1)$ is finitely generated. Write $V_1$ for the component of $\tilde{X} \setminus C_1$ containing $U_1$. Since $X$ is Tucker, $\pi_1(V_1)$ is finitely generated. But Van Kampen's theorem implies that $\pi_1(V_1)$ is equal to the free product of $\pi_1(U_1)$ with a finitely generated free group (possibly trivial). Hence, by Grushko's theorem, $\pi_1(U_1)$ is finitely generated, as desired. □

We will now apply the above result to free products with amalgamations and HNN extensions. We give the details for amalgamations. The case for HNN extensions is similar.

Suppose $G = A *_C B$, with $A$ and $B$ finitely presented and $C$ finitely generated. Take spaces $K_A$ and $K_B$ with fundamental groups $A$ and $B$, respectively. By attaching mapping cylinders from a bouquet of circles, one gets a space $X$ with fundamental group $G$, and a two-sided subcomplex $S$ (the bouquet of circles), for which $X_S$ is $\pi_1$-injective in $X$. Observe that $S$ does not have fundamental group being $C$, and $S$ is not itself $\pi_1$-injective in $X$. However, we may still use theorem 4.1, and we get the following result:

**Corollary 4.2.** *Suppose $A$ and $B$ are finitely presented and $C$ is finitely generated.*

1. *$A *_C B$ is Tucker iff both $A$ and $B$ are Tucker.*
2. *$A *_C \phi$ is Tucker iff $A$ is Tucker.*

Now, just as in [2], we may apply the HNN extension approach to one-relator groups which yields:

**Corollary 4.3.** *All one-relator groups are Tucker.*

*Proof.* Suppose $G$ is a one-relator group. By a result in [4], there is a finite sequence of finitely generated groups $H_1, H_2, \ldots, H_n = G$ where, for each $i < n$, either $H_{i+1}$ or $H_{i+1} * Z$ is an HNN-extension of $H_i$ over a finitely generated group and such that $H_1$ is either a free group or a free product of a free group and a finite cyclic group. The result now follows from corollary 4.2. □

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