MA126 Calculus II

Exam 3

Name: Solutions

There are 6 questions worth a total of 48 points. For full credit, show all steps of your work. A calculator may not be used.

1. Apply the integral test to show whether \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) converges or diverges. (The integral test may be applied in this case since \( f(x) = \frac{1}{\sqrt{x}} \) is a continuous, positive, decreasing function of \( x \).) (7 points)

\[
\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \text{ with } \int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx
\]

\[
\int_{1}^{\infty} \frac{1}{\sqrt{x}} \, dx = \lim_{b \to \infty} \int_{1}^{b} x^{-\frac{1}{2}} \, dx
\]

\[
= \lim_{b \to \infty} \left[ 2x^{\frac{1}{2}} \right]_{1}^{b}
\]

\[
= \lim_{b \to \infty} \left( 2b^{\frac{1}{2}} - 2 \right)
\]

So \( \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \) diverges.

2. Apply the root test to show whether the following infinite series converges or diverges. (The root test may be applied in this case since the terms are non-negative.) (7 points)

\[
\sum_{n=1}^{\infty} \frac{7}{(2 + \frac{1}{n})^{3n}}
\]

\[
\lim_{n \to \infty} \left( \frac{n\sqrt[3n]{7}}{\sqrt[3n]{(2 + \frac{1}{n})^{3n}}} \right) = \lim_{n \to \infty} \frac{n\sqrt[3n]{7}}{(2 + \frac{1}{n})^{3}}
\]

\[
= \frac{1}{2^3} = \frac{1}{8} < 1 \Rightarrow \text{series convergent}
\]

* by Theorem 5 on page 492 (#3)
3. The following infinite series have positive terms that are decreasing functions of \( n \). Determine whether each of these series converge or diverge. Show your work or write the name of the test that you are applying. (15 points)

a. \( \sum_{n=4}^{\infty} \frac{n^2}{n^2-3n} \)

**n-th term test:** \( \lim_{n \to \infty} \frac{n^2}{n^2-3n} \neq 0 \Rightarrow \text{diverges} \)

\[ \lim_{n \to \infty} \frac{n^2}{n^2-3n} = 1 \]

b. \( \sum_{n=1}^{\infty} \left( \frac{1}{10} \right)^n \)

**geometric series test:**

\( \sum_{n=1}^{\infty} r^n \) converges if \( |r| < 1 \)

and \( \left| \frac{1}{10} \right| < 1 \)

(also \( n-th \) root test or ratio test easy to apply here)

c. \( \sum_{n=8}^{\infty} \frac{1}{n-7} \)

**Diverges by the Comparison Test:**

\( \frac{1}{n-7} > \frac{1}{n} \) since \( n-7 < n \)

but \( \sum_{n=8}^{\infty} \frac{1}{n} \) diverges so \( \sum_{n=8}^{\infty} \frac{1}{n-7} \) must diverge too

(d) \( \sum_{n=1}^{\infty} e^{-n} \)

**geometric series test:**

\( r = \frac{1}{e} < 1 \) \( \Rightarrow \) converges

n-th root test:

\( \lim_{n \to \infty} \sqrt[n]{e^{-n}} = e^{-1} = \frac{1}{e} < 1 \)

\( \Rightarrow \) converges

**ratio test:**

\( \lim_{n \to \infty} \frac{e^{-n+1}}{e^{-n}} = e^{-1} < 1 \)

\( \Rightarrow \) converges

d. \( \sum_{n=1}^{\infty} \frac{(n+1)^2}{(n-1)!} \)

**Ratio test:**

\[ \lim_{n \to \infty} \frac{(n+2)^2}{n!} \cdot \frac{n!}{(n-1)!} = \frac{(n+2)^2}{n(n+1)^2} = \frac{n^2+4n+4}{n^3+2n^2+2n} \]

\[ = \lim_{n \to \infty} \frac{1\frac{n}{n} + \frac{4}{n^2} + \frac{4}{n^3}}{1 + \frac{2}{n} + \frac{1}{n^2}} = 0 < 1 \Rightarrow \text{converges} \]
4. The following alternating series correspond to the infinite series of the previous problem.

Indicate whether each alternating series converges absolutely, converge conditionally, or diverges by checking the appropriate boxes. You do not need to show any work for this problem! (5 points)

<table>
<thead>
<tr>
<th>Series</th>
<th>Converges Absolutely</th>
<th>Converges Conditionally</th>
<th>Diverges</th>
</tr>
</thead>
<tbody>
<tr>
<td>a. $\sum_{n=4}^{\infty} (-1)^n \frac{n^2}{n^2 - 3n}$</td>
<td>□</td>
<td>□</td>
<td>✗#</td>
</tr>
<tr>
<td>b. $\sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{10}\right)^n$</td>
<td>✗*</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>c. $\sum_{n=8}^{\infty} (-1)^n \frac{1}{n-7}$</td>
<td>□</td>
<td>✗</td>
<td>□</td>
</tr>
<tr>
<td>d. $\sum_{n=1}^{\infty} (-1)^n e^{-n}$</td>
<td>✗*</td>
<td>□</td>
<td>□</td>
</tr>
<tr>
<td>e. $\sum_{n=1}^{\infty} (-1)^n \frac{(n+1)^2}{(n+1)!}$</td>
<td>✗*</td>
<td>□</td>
<td>□</td>
</tr>
</tbody>
</table>

* The series in Problem #3 have positive terms, so if they converge, they converge absolutely.

# In Problem 3, we showed that b>d and e converge so here we select 'converges absolutely'.

! A series that fails, the nth term test diverges.

* A series that fails to converge absolutely but which passes the nth term test may converge conditionally if the terms are decreasing.

Review: Alternating Series Test
Absolute Convergence Test
Nth Term Test
5. Consider the power series \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{10^n} \). (a) Find the radius and interval of convergence. For what values of \( x \) does the series converge (b) absolutely, (c) conditionally? (7 points)

Apply ratio test to find interval of
absolute convergence: 
\[
\left| \frac{(x-2)^{n+1}}{10^{n+1}} \cdot \frac{10^n}{(x-2)^n} \right| = \left| \frac{x-2}{10} \right| < 1
\]

\[-1 < \frac{x-2}{10} < 1\]
\[-10 < x-2 < 10\]
\[-8 < x < 12\]

we'll have absolute convergence on this interval by Thm 18

check endpoints:
@ \(-8\): \( \sum \frac{(-10)^n}{10^n} = \sum -1 \) these diverge by nth-term test
@ 12: \( \sum \frac{10^n}{10^n} = \sum 1 \)
so there is no conditional convergence

The radius is 10 since
\((-8, 12) = (2-10, 2+10)\)

(a) interval: \((-8, 12)\)
(b) \(-8 < x < 12\)
(c) no values of \( x \)

6. The Taylor Series generated by \( f \) centered at \( x=a \) is \( \sum_{k=0}^{\infty} \frac{f^{k}(a)}{k!} (x - a)^k \). Compute the Taylor Series for \( f(x) = x^3 + 2x - 12 \) centered at the point \( a = 2 \). (7 points)

\[
f(x) = x^3 + 2x - 12 \quad f(2) = 0
\]
\[
f'(x) = 3x^2 + 2 \quad f'(2) = 14
\]
\[
f''(x) = 6x \quad f''(2) = 12
\]
\[
f'''(x) = 6 \quad f'''(2) = 6
\]
\[
f^{(n)}(x) = 0 \quad \text{for } n > 3
\]
\[
f^{(n)}(2) = 0 \quad \text{for } n > 3
\]

So
\[
f(x) = \frac{0(x-2)^0}{0!} + \frac{14(x-2)^1}{1!} + \frac{12(x-2)^2}{2!} + \frac{6(x-2)^3}{3!}
\]

\[
= 14(x-2) + 6(x-2)^2 + (x-2)^3
\]