Categories for Knotted Curves, Surfaces and Quandles

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Abstract. This paper is an overview of categorical structures that are associated to embedded surfaces in 3-space, generic surfaces in 3-space, and knotted surfaces in 4-space. The paper is short on technical details, but replete with descriptions and pictures. The motivation is to give a sense in which topological phenomena can be algebratized, and how invariants can be constructed from such translations. Many steps have been completed in such a program. However, there is much that remains to be done.

1 Introduction

The discovery of the Jones polynomial lead to a broad research area that is called Quantum Topology since it combines ideas from quantum mechanics and elementary knot theory. Throughout this development, category theory has played an essential role. The diagrammatical nature of the Jones polynomial lead to the notion of a braided monoidal category [23]. Functors from the category of tangles to representations of quantum groups lead to generalizations of the Jones polynomial such as the colored Jones polynomial, and non-associative tangle theory was used [8] for finite type invariants via Kontsevich integrals. The representations at roots of unity were used to define quantum 3-manifold invariants [43] that are a mathematical definition of Witten's [47] invariants. Topological Quantum Field Theories (TQFT) as axiomatized by [2] are formed as functors from a cobordism category to a category of Hilbert spaces for example, and a variety of TQFT have been constructed (see for example, [25, 38, 46]). The literature on these developments is extensive; for example, [49] develops the theory from the point of view of the category of tangles, [38] investigates TQFTs on 3-manifolds, and [45] gives a particularly detailed and clear exposition of the Reshetikhin-Turaev [43] and Turaev-Viro [46] invariants (see also [14]). By using the Kauffman bracket definition of the Jones polynomial, Khovanov has been able to produce an homology theory in which the Jones polynomial can be interpreted as a graded Euler characteristic. His approach was motivated by categorifying the representations of $U_q(sl_2)$.

Since the discovery of the Jones polynomial, it has been questioned whether higher dimensional analogues exist. That is, "Can invariants of higher dimensional knottings be defined via diagrammatic methods?" As it was successful to investigate category theoretical structures of tangles and 3-manifolds and their functors in constructing generalizations of the invariants for knots and 3-manifolds, one of the approaches to higher dimensions was to investigate category theoretical structures for knots and manifolds in higher dimensions, in particular for knotted surfaces in 4-space [18, 5, 27, 37] and 4-manifolds [16, 19, 21, 20]. In these approaches, categorifications of algebraic systems and 2-categories have been investigated that effectively represent surfaces and 4-manifolds. Categorifications and 2-categories have been investigated for a long time from purely algebraic points of view and motivations as higher dimensional algebras (for example, [10, 33] and [6] through [3]).

These notions motivated my collaborators and me to define invariants via quandle cohomology theory [15]. The notion was defined as a modification of rack homology theory [30], and the quandle cocycle invariant was constructed [15] in a state-sum form. Some aspects of quandles and their knot invariants have been studied from algebraic [1, 26, 28] and category theoretical [3, 22] points of view. The purpose of this paper is to revisit category theoretical aspects of surfaces, quandles and their cohomology theories and invariants, from a point of view with these recent developments in mind. In Section 2, I revisit categories related to surfaces, and in Section 3, I review the definition of the quandle cocycle invariants.

2 Categories for Surfaces

2.1 Surfaces embedded in 3-space and non-associativity

The idea of a 2-category can be exemplified by embedded surfaces in 3-space, generic surfaces in 3-space, and knotted surfaces in 4-space. In this section, I develop these three 2-categories from the ground up. It is my hope that such examples will indicate clearly the ideas of objects, morphisms, 2-morphisms, relations among these, and the so-called 4-square relation. From my provincial point of view (certainly incorrect), the notion of a 2-category is invented to study surfaces. But more generally, the result of Fisher [27], and more rigorously Baez-Langford [5], indicates that the 2-category of knotted surfaces is a free 2-category of a very specific type — a free braided monoidal 2-category with duals on an

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unframed self-dual object. Thus it provides evidence for the tangle hypothesis, and invariants of knotted surfaces can be found as representations into another braided monoidal 2-category with the same structures.

Associators. We construct a 2-category $\mathcal{EMB}_0$ based upon embedded arcs and surfaces bounded by these. Consider the set of k-fold subsets, $S \subset 2^{\mathbb{N}}$ of the natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$. A typical object in $\mathcal{EMB}_0$ is a subset $\{i_1 < i_2 < \cdots < i_k\} \subset \mathbb{N}$. If $k = 0$, such a subset is the empty set. A groupoid is constructed as a collection of morphisms between such objects. Here *groupoid* means a set upon which an associative binary operation is partially defined. That is the composition ab may or may not be defined. Alternatively, a groupoid is a category in which the collection of objects forms a set.

For $j = 1, \ldots, k$, if $i_{j+1} \neq i_j + 1$ there is a morphism

$$|_{j-1} \nearrow |_{k-j-1} : \{i_1 < i_2 < \cdots < i_j < \cdots < i_k\}$$

$$\to \{i_1 < i_2 < \cdots < i_{j-1} < i_j + 1 < \cdots < i_k\}.$$

Under analogous circumstances there is a morphism back:

$$|_{j-1} \nwarrow |_{k-j-1} : \{i_1 < i_2 < \cdots < i_{j-1} < i_j + 1 < \cdots < i_k\}$$

$$\to \{i_1 < i_2 < \cdots < i_j < \cdots < i_k\}.$$

If $i_{j+1} = i_j + 1$, then there is a morphism,

$$|_{j-1} \cap_j |_{k-j-2} : \{i_1 < i_2 < \cdots i_j < i_j + 1 < \cdots < i_k\}$$

$$\to \{i_1 < i_2 < \cdots i_{j-1} < i_{j+2} < \cdots < i_k\}.$$

Similarly, when $i_{j+1} - i_j > 2$, there is a morphism:

$$|_{j-1} \cup_{i_j+1} |_{k-j-2} : \{i_1 < i_2 < \cdots i_j < i_{j+1} < \cdots < i_k\}$$

$$\to \{i_1 < i_2 < \cdots i_j < i_{j'} < i_{j'+1} < i_{j+1} < \cdots < i_k\}$$

where the symbol j' does not carry any meaning until the resulting set is reindexed.

There is a tensor product of two subsets A and B that is obtained by shifting B (if necessary) to a set B' that is completely to the right of A, and juxtaposing A and B. Similarly morphisms may be tensored. Thus the morphism $|_{j-1} \nearrow_j |_{k-j-1}$ is strictly speaking the tensor product of an identity morphism on a $(j-1)$-element subset, a northeastern morphism on one object, and another identity morphism.

At the level of objects the structure is hardly associative. Indeed, the initial shift of the object B can affect the grouping of the resulting object. The next paragraphs contain more details of the lack of associativities in the tensor product.

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There is a category whose objects are finite elements in $2^{\mathbb{N}}$ and whose morphisms are generated by $\cup$, $\cap$, $|$, $\nearrow$ and $\nwarrow$. The identity object is the empty set, the identity morphism on $\{i_1 < i_2 < \ldots < i_k\}$ is the tensor product of $|$; specifically $|_{i_1, i_2, \ldots, i_k}$. The geometric depiction of the five generating morphisms is given in Fig. 1.

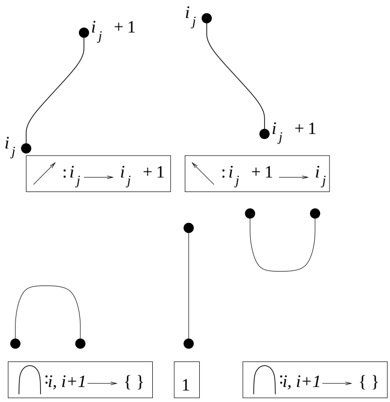

Figure 1: The generating 1-morphisms in the 2-category $\mathcal{EMB}_0$

The objects in this category determine partially associated products of indeterminants as follows: A given set $\{i_1 < i_2 < \ldots < i_k\}$ has its elements associated by their proximity to each other. Thus elements that are closer together are viewed as having been associated. For example, $\{1, 2, 4, 7\}$ is related to the associated product $((ab)c)d$ since the first two elements are closest and since 4 is closer to 2 than it is to 7. If a subset consist of elements in an arithmetic progression, then these element can be treated as a partially parenthesized product. For example, the set $\{1, 2, 3, 4, 5, 8, 10, 12\}$ is determines the partially parenthesized product $(x_1 x_2 x_3 x_4 x_5)(x_8 x_{10} x_{12})$, and so in this case, we assume that the products, $x_1 x_2 x_3 x_4 x_5$ and $x_8 x_{10} x_{12}$ are well-defined. Once an un-associated product is presented, then we treat it as a single entity.

More rigorously, to a given set $S = \{i_1 < i_2 < \ldots < i_k\}$ there is a corresponding monomial in non-associative variables $x_{i_1}, \ldots, x_{i_k}$. There is an association of these variable that is determined by the proximity of their subscripts. To determine the association compute the minimum among the differences $i_{j+1} - i_j$. If j achieves the minimum value, then parenthesize $(x_j x_{j+1})$. If the sequence $i_j, i_{j+1}, \ldots i_{j+\ell}$ is in arithmetic progression and this achieves the minimum difference, then group all of the corresponding variables together. By induction, an association scheme can be constructed. Here is one further example: $\{1, 3, 4, 5, 7, 10\}$ corresponds to $(x_1(x_3 x_4 x_5)x_7)x_{10}$. The products $(x_3 x_4 x_5)$ and $x_1(x_3 x_4 x_5)x_7$ are only partially associated since the corresponding subscripts are equally close.

The correspondence between a given subset and a partially parenthesized

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product is well-defined, but different subsets can determine the same product. It is possible to choose standard representatives of each partially parenthesized product by rescaling the subset.

We can think of a pair of subsets, $\{i_1 < i_2 < \ldots < i_k\}$ and $\{j_1 < \ldots < j_{k'}\}$ where $i_k < j_1$ as being a distinct entity from $\{i_1 < \ldots i_k < j_1 < \ldots < j_{k'}\}$. Geometrically, to make this distinction we can add an increment to each element of the latter set so that there is a parenthetical insertion forced between the two sets. To exemplify these ideas consider $\{1, 2, 4, 7\}$ and $\{1, 4, 6, 7\}$. The tensor product is the set $\{1, 2, 4, 7, 8, 11, 13, 14\}$. This corresponds to the parenthesized string $((x_1x_2)x_4)(x_7x_8)(x_{11}(x_{13}x_{14}))$. Whereas $\{1, 2, 4, 7, 11, 14, 16, 17\}$ corresponds to $(((x_1x_2)x_4)x_7)(x_{11}(x_{14}(x_{16}x_{17})))$. Thus the structure of the 1-category at this level is not monoidal.

If we want to shift $\{j, j+1, \ldots, j+k\}$ right one unit to the set $\{j+1, j+2, \ldots, j+k+1\}$, the topography of the morphism $\nearrow$ allows all of these points to be moved simultaneously. Thus $\nearrow_{\{1,\ldots,k\}}$ denotes the tensor product of $\nearrow_1, \ldots, \nearrow_k$. And right shifts to $\{j, j+1, \ldots, j+k\}$ can be achieved using the same scheme.

The process of moving elements from one subset to another by shifting them individually achieves a method of re-associating the objects. In this manner, the morphisms $\nearrow$ and $\nwarrow$ are associators. Thus the 1-category, $\mathcal{EMB}_0$, is premonoidal provided the following two transformations (and their variants) of the set $\{1, 2, 4, 7\} \to \{1, 4, 6, 7\}$ given as

$$\begin{aligned}
&|_1 \nearrow_3 |_{6,7}(|_1 \nearrow_2 |_{6,7}(|_{1,2} \nearrow_5 |_7(|_{1,2} \nearrow_4 |_7(\{1, 2, 4, 7\}))))\\
&= |_1 \nearrow_3 |_{6,7}(|_1 \nearrow_2 |_{6,7}(|_{1,2} \nearrow_5 |_7(\{1, 2, 5, 7\})))\\
&= |_1 \nearrow_3 |_{6,7}(|_1 \nearrow_2 |_{6,7}(\{1, 2, 6, 7\}))\\
&= |_1 \nearrow_3 |_{6,7}(\{1, 3, 6, 7\})\\
&= \{1, 4, 6, 7\}
\end{aligned}$$

and

$$\begin{aligned}
&|_{1,4} \nearrow_5 |_7(|_1 \nearrow_{3,4} |_7(|_1 \nearrow_2 |_{4,7}(\{1, 2, 4, 7\})))\\
&= |_{1,4} \nearrow_5 |_7(|_1 \nearrow_{3,4} |_7(\{1, 3, 4, 7\}))\\
&= |_{1,4} \nearrow_5 |_7(\{1, 4, 5, 7\})\\
&= \{1, 4, 6, 7\}
\end{aligned}$$

are equal. Note that at this stage, only objects and morphisms have been defined. If we want a 1-category, then we need to assert identities among 1-morphisms. So if we want a premonoidal 1-category with objects the finite elements of $2^{\mathbb{N}}$ and with morphisms $\cup$, $\cap$, $\nearrow$, $\nwarrow$, and $|$, then we would assert (among other things) that these two transformations are equal. The transformations are encoding the pentagon relation.

However, in the 2-category, we might assert the existence of a 2-morphism $\phi_{1,2,4,7}$ such that

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$$\phi_{1,2,4,7}(|_{1,4} \nearrow_5 |_7) \circ (|_1 \nearrow_{3,4} |_7) \circ (|_1 \nearrow_2 |_{4,7})$$

$$= (|_1 \nearrow_3 |_{6,7}) \circ (|_1 \nearrow_2 |_{6,7}) \circ (|_{1,2} \nearrow_5 |_7) \circ (|_{1,2} \nearrow_4 |_7)$$

The map $\phi_{1,2,4,7}$ is called the *pentagonator*. An inspection of the superficial diagram of the pentagonator 2 reveals that it can be decomposed in terms of a more simple operation. Each arrow $\nearrow$ (or $\nwarrow$) is depicted as a neighborhood of an inflection point in its diagrammatic representation. Distant inflection points commute. And thus we can assert the existence of a more basic invertible 2-morphism, R, that represents the commutation of distant inflection points and that satisfies a Yang-Baxter type relation. In Baez and Langford [5], this morphism is a tensorator between the pair of distant arrows.

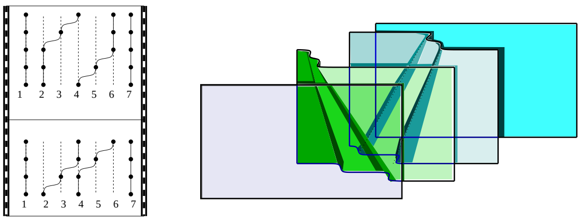

Figure 2: A surface that envelops the pentagonator

Observation. *The pentagonator can be written in terms of a tensorator.*
Proof.

$$R_{2,4}R_{3,5}R_{2,4}(|_{1,4} \nearrow_5 |_7) \circ (|_1 \nearrow_{3,4} |_7) \circ (|_1 \nearrow_2 |_{4,7})$$

$$= R_{2,4}R_{3,5}R_{2,4}(|_{1,4} \nearrow_5 |_7) \circ (|_{1,5} \nearrow_3 |_7) \circ (|_{1,2} \nearrow_4 |_7) \circ (|_1 \nearrow_2 |_{4,7})$$

$$= R_{2,4}R_{3,5}(|_{1,4} \nearrow_5 |_7) \circ (|_1 \nearrow_3 |_{5,7}) \circ (|_{1,4} \nearrow_2 |_7) \circ (|_{1,2} \nearrow_4 |_7)$$

$$= R_{2,4}(|_1 \nearrow_3 |_{6,7}) \circ (|_{1,3} \nearrow_4 |_7) \circ (|_1 \nearrow_2 |_{5,7}) \circ (|_{1,2} \nearrow_4 |_7)$$

$$= (|_1 \nearrow_3 |_{6,7}) \circ (|_1 \nearrow_2 |_{6,7}) \circ (|_{1,2} \nearrow_5 |_7) \circ (|_{1,2} \nearrow_4 |_7)$$

See also Fig. 3. This completes the proof.

The pentagonator is not, strictly speaking, restricted to acting on 1-morphisms of subset of size 4. A given partially associated product, $((AB)C)D$ with four constituents, $A = \{i_1 < i_2 < \ldots i_j\}$ and $(i_{k+1} - i_k)$ constant for $k = 1, \ldots j$ (similar descriptions for B, C, etc.) can be reassociated by similar inflection points (literally $\nearrow_B$ and so forth), and the resulting pentagonator is a "pleated multi-surface" in which the pleats commute past each other via corresponding Yang-Baxterators.

In addition to Yang-Baxterators between distant (and therefore commuting inflection points), we assert the existence of Yang-Baxterators between commuting critical points. (Baez and Langford treat all such 2-morphisms as tensorators).

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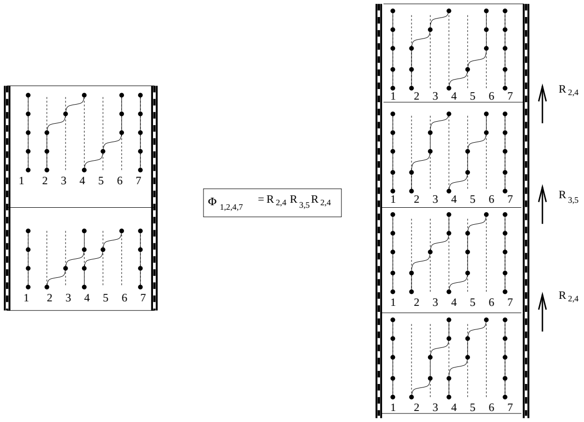

Figure 3: Factoring the pentagonator via Yang-Baxterators

Thus the compositions $(\cap_{1,2}) \circ (|_{1,2} \cap_{3,4})$ and $(\cap_{3,4}) \circ (\cap_{1,2}|_{3,4})$ are related by a 2-morphism (also called R) and other commutations of critical points or inflection points are given by similar 2-morphisms.

Further contrasts in the 1-category and the 2-category. At the level of a 1-category, one may wish to assert an identity among 1-morphisms such as $\cup_{1,2} \circ \cap_{1,2} = |_{1,2}$. The geometric reason is that a given set has a pair of successive elements annihilated by means of a $\cap$, and then the pair is recreated via a $\cup$. In a physical analogy, a pair of particles annihilate each other, emit a photon, and then the photon splits into a pair of identical particles. If the emission of the photon is so short lived that it can't be detected, then we would not want to include its life in a measurement.

Similarly, we may want to assert an identity between $(\cap_{1,2}|_3) \circ (|_1 \cup_{2,3})$ and $|_1$. Such identities have obvious analogues that are dependent upon shifting indices, and the second identity has an analogue obtained by turning the diagram for $(\cap_{1,2}|_3) \circ (|_1 \cup_{2,3})$ upside-down.

In a rigid monoidal category with duals, one asserts that $(\cap_{1,2}|_3) \circ (|_1 \cup_{2,3}) = |$. (In the premonoidal case, the equality might be written as $(\cap_{1,2}|_3) \circ (|_1 \cup_{2,3}) = \nearrow$). In this context, there is a self-dual object generator V. The maps $\cup$ and $\cap$ represent respectively represent a double map $\cup : 1 \to V \otimes V$, and an evaluation map $\cap : V \otimes V \to 1$. Here 1 represents the identity object (the empty set), and V represents a single point. In [42] associators are given to axiomatize a monoidal category with duals, but therein there are no diagrammatic representations given of associators are given.

In the current 2-category, we assert the existence of 2-morphisms: $S : \cup_{1,2} \circ \cap_{1,2} \Rightarrow |_{1,2}$ — a saddle point, $C : (\cap_{1,2}|_3) \circ (|_1 \cup_{2,3}) \Rightarrow |_1$— a cusp (called a triangulator in [5]), $B : \emptyset \Rightarrow \cap \circ \cup$ — a birth of a simple closed curve, and

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$D : \cap \circ \cup \Rightarrow \emptyset$ — a death of a simple closed curve. These 2-morphisms are depicted via their diagrams on the top row of Fig. 4. We note here that the rest of the figure refers to the next subsection.

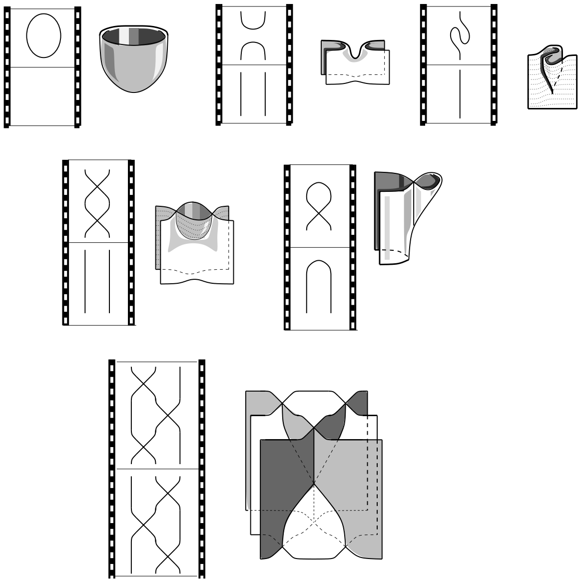

Figure 4: The 2-morphisms in the category of immersed surfaces

Furthermore, one asserts the existence of the two 2-morphisms that are depicted in Fig. 5. The first represents a 2-morphism bend $:\nwarrow \circ \nearrow :\Rightarrow |$, and the second represents a 2-morphisms curve : $\cap_{1,2} \circ \nearrow_{1,2} \Rightarrow \cap_{1,2}$. The former replaces the assumption that $\nearrow$ and $\nwarrow$ are inverse morphisms, and the latter replaces the assumption that the location of the cap is immaterial.

The generating objects and morphisms in the 2-category $\mathcal{EMB}_0$. The objects in the category are finite subsets of the positive integers. The generating 1-morphisms are $\nearrow$, $\nwarrow$, $|$, $\cup$, and $\cap$. The set of morphisms between two subsets $\{i_1, \ldots, i_k\}$ and $\{j_1, \ldots, j_{k'}\}$ is empty if $k \equiv k' + 1 \pmod 2$, otherwise, it consists of the set of diagrams of disjoint arcs that inner-connect the points in the subsets. An arc that connects a point i_ℓ at the bottom to j'_ℓ at the top, has the opportunity to meander to the left or right, up or down before it makes the connection. We can decompose this meandering in terms of the generating morphisms by choosing suitable height functions. The generating 2-morphisms consist of births, B, and deaths, D, of simple closed curves ($\emptyset \Rightarrow \cap \circ \cup$ or $\cap \circ \cup \Rightarrow \emptyset$), saddles, S, ($\cup \circ \cap \Rightarrow ||$ and vice-versa), cusps, C ($(\cap \otimes |) \circ (| \otimes \cup) \Rightarrow |$ and vice-versa), tensorators, bends, curves and their inverses.

A topological/categorical question. In the current set up, suppose that two

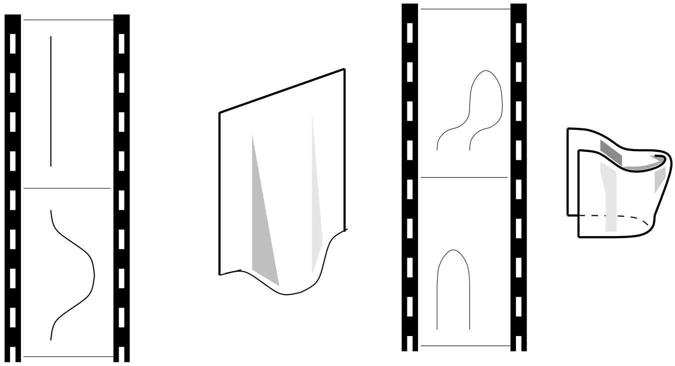

Figure 5: Extra 2-morphisms in the pre-monoidal setting

1-morphisms both with source $A = \{i_1, \ldots, i_k\}$ and target $B = \{j_1, \ldots, j_\ell\}$. are given. An uncategorical question to ask would be, "Are these 1-morphisms equal?" It is uncategorical, since we have been reminded that in higher category theory, that it is undesirable and unnatural to consider different things to be equal. Instead we can ask, "Is there a 2-morphism connecting them?" To ask this becomes a topological/geometric question. We have two pairs of fixed arcs meandering and inter-connecting among the specific subsets A and B. The topological question is if there is a surface with boundary that interpolates between the two arcs and if that surface can be decomposed in terms of the generating 2-morphisms that have been written down.

To address this question, suppose there is a surface, project it to a plane (called the retinal plane below), and track critical arcs and arcs of inflection points in the retinal plane. I expect that a Cerf theory type argument like that given in [17] will show that these 2-morphisms suffice to connect the two 1-morphisms. In general, it is not hard to construct a surface that interpolates between the pair of 1-morphisms. The first question is whether or not such a surface can be decomposed in terms of our generating 2-morphisms.

If so, then we ask if there is a 2-morphism that connects these two 1-morphisms that does not factor with any saddle points maxima or minima. Clearly, the number of arcs that start and end at either A, B, or travel between them has to be the same for each 1-morphism. The reason for excluding the critical points is that these are precisely the 2-morphisms that we do not want to consider to be invertible. Thus the categorical question is, "Are the two 1-morphisms equivalent in the sense that they are related by a sequence of invertible 2-morphisms?"

Composition of 2-morphisms. In $\mathcal{EMB}_0$ or any of our subsequent 2-categories, 1-morphisms are composed by juxtaposing vertically. In my conventions, the com-

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position $f \circ g$ is represented with f above and g below. If f and g were functions, then the domain of the function would be represented at the bottom of the diagram, g would be applied first, and then f would be applied. The advantage is that when reading from left to right we can draw from top to bottom. In the abstract tensor notation of Kauffman's work [34], this means that the matrices represented by the diagrams are applied to column vectors that appear on the right of an equation.

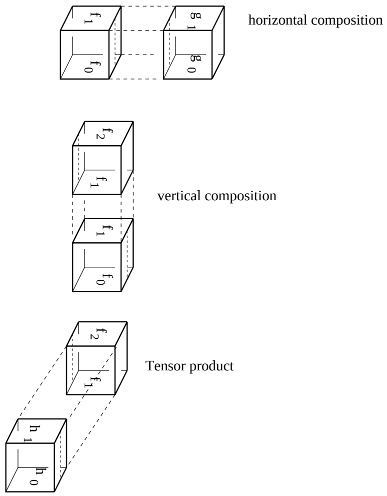

Figure 6: Compositions of 2-morphisms

In any 2-category, there are vertical and horizontal composition of 2-morphisms. The horizontal composition of two morphisms $F : f_0 \Rightarrow f_1$, and $G : g_0 \Rightarrow g_1$ is defined when the sources of the arrows g_0 and g_1 coincide with the targets of the arrows f_0 and f_1. In our movie description of the 2-morphisms, successive stills in the movie differ by a composition of 2-morphism. To compose two 2-morphisms horizontally, we create two movies representing the 2-morphisms. The target of the 1-morphisms f_0 and f_1 consists of, say, m points and these are the sources of the 1-morphisms g_0 and g_1. We can compose the two 2-morphisms by super-imposing the two changes in scenes that are represented by F and G. In other words, both F and G are represented by surfaces with boundary, and the right boundary of F is glued to the left boundary of G. The vertical composition of two 2-morphisms is represented by the succession of stills.

To be more specific, Suppose that the source 1-morphism $\hat{n}$ of f_0 is depicted on the left bottom edge of a box that contains the surface representing the 2-morphism F. The left to right axis on the 1-morphism diagram translates to the back to front axis (x-axis) on the box. The left face (xz-plane, $y = 0$) of the box contains the arcs $\hat{n} \times I$. The right face of the box contains the arcs, $\hat{m} \times I$, that

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consist of the targets of f_0 and f_1 multiplied by an interval factor. The bottom face (xy-plane $z = 0$) contains the diagram for f_0. The top face ($z = 1$) contains the diagram for f_1. The two sides of the box, $x = 0$ and $x = 1$, are empty. The description of the box holding a 2-morphism, G, is similar. Now the vertical composition of F and G is obtained by stacking G on top of F and rescaling the vertical dimension of the box. The horizontal composition is obtained by gluing the boxes together with F on the left and G to the right. Meanwhile, the tensor product of 2-morphisms is obtained by juxtaposing the boxes in the x direction. Figure 6 indicates the various compositions. Observe that the 4-square relation is a consequence of the fact that these four boxes can be stacked two on top of two, or two to the right of a stack of two.

A further simplification. To finish specifying the 2-category, we need to specify relations among the 2-morphisms. The situation is getting complicated, because heretofore we have assumed a pre-monoidal structure. Henceforth, an object $\hat{k}$ will be any set of numbers with k elements where $k = 0, 1, \ldots$. We depict such a set as $\hat{k} = \{1, \ldots, k\}$, and the case $k = 0$ corresponds to the empty set. However, the location of the points on the line is assumed to be immaterial. Thus we abandon the need to specify left-to-right ($\nearrow$) or right-to-left ($\nwarrow$) motion in the diagrams for the 1-morphisms. The tensor product $\hat{k} \otimes \hat{k'} = \widehat{k + k'}$ is strictly associative, and any arc that has no critical points is assumed to be the identity. Otherwise, arcs are assumed to have generic critical points (x^3 is excluded for example). The generating 1-morphisms are $\cup$, $\cap$, and $|$. The generating 2-morphisms are B, D, S, C, and tensorators— those that involve the commutation of distant critical points. In this 2-category $\mathcal{EMB}$, we will describe the relations among 2-morphisms. The pre-monoidal structure has been adjusted to be monoidal.

The cusp (triangulator) 2-morphism is invertible. There are two types of cusps: those that create a pair of critical points, and those that cancel a pair of critical points. Thus there are two possible vertical compositions of cusps. The terminology from singularity theory is *lips* or *beak-to-beak*. A saddle point can cancel with a local maximum or minimum. There is a *swallow-tail* cancelation of a pair of cusps in the presence of a Yang-Baxterator for the critical points. Finally, a cusp can be turned upside-down in a the presence of a saddle point. All of these relations can be easily depicted in terms of the fold lines and cusps that appear on the projection of embedded surfaces in 3-space. These relations are depicted in Figs. 7, 8, 11, and 9. Observe that the right had parabolic cylinder is unnecessary to the relationship — it is drawn only to anchor the idea.

In a rigid monoidal 1-category with duals, the maps $\cup$ and $\cap$ satisfy the relation $(\cap \otimes |) \circ (| \otimes \cup) = |$. In the corresponding 2-category we relax this condition to the existence of a 2-isomorphism $(\cap \otimes |) \circ (| \otimes \cup) \Rightarrow |$. This 2-morphism is depicted in the graphical notation as a cusp on the projection of a surface. The fact that it is a 2-isomorphism is depicted in Fig. 11; the figure indicates that the two movies representing the vertical composition of 2-morphisms represent the same surface. Or in categorical language there is a commuting polytope of the

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corresponding 2-morphism. Another relation that is imposed upon the duality 2-morphism by the geometry is the swallowtail relation that is depicted in Fig. 8. In general, a full list of relations among 2-morphisms consist of the following:
Relations among the 2-morphisms.

- commuting distant 2-morphisms;
- Yang-Baxter relations among nearby tensorators;
- the cancelation or introduction of a pair of similar such tensorators ($ab \Rightarrow ba \Rightarrow ab$)
- the cancelation or introduction of a saddle point and a birth or death (Fig. 7);
- lips cancelations or introduction of cusps (Fig. 11);
- beak-to-beak cancelation or introduction of cusps (Fig. 11);
- swallow-tail cancelation or introduction of cusps (Fig. 8);
- interchanging an upward cusp and a saddle point with a saddle point and a downward cusp (Fig. 9).

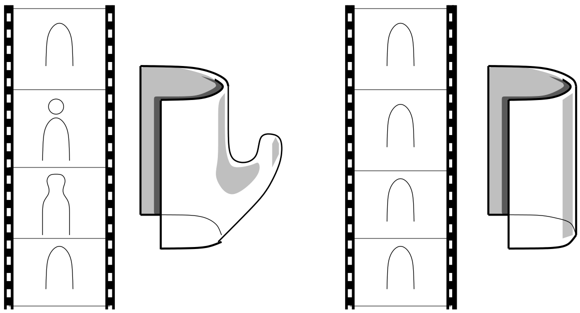

Figure 7: The cancelation of a maximum point and a saddle point

Now all of these relations can be written in purely categorical language. See [5] for example. Moreover each is quantified as a codimension 1 singularity between surface maps, and each such relation can be thought of kinematically. I encourage you to determine if the free monoidal 3-category with duals on one self dual object generator consists of the non-negative integers as objects with addition of integers as the monoidal structure, 1-morphisms generated by $\cup$, $\cap$ and $|$. Generating 2-morphisms would be given as cusps, saddles, births, and deaths. And generating 3-morphisms would be the singularities above, births and

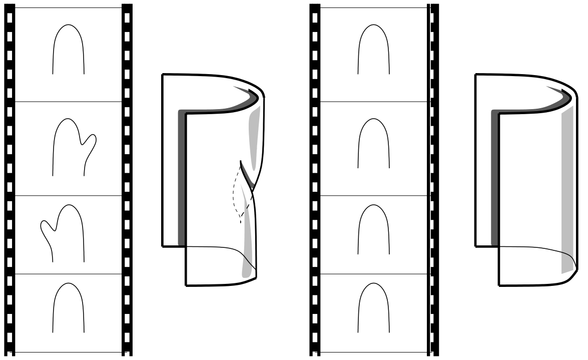

Figure 8: The swallow-tail relation among 2-morphisms

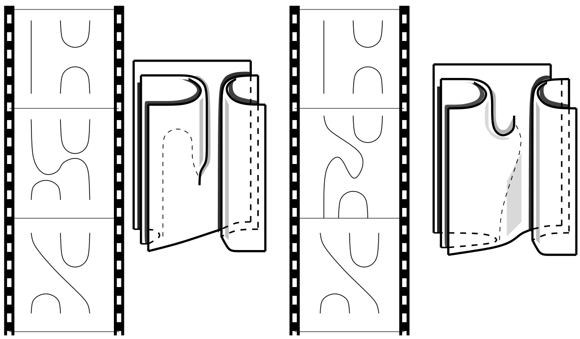

Figure 9: The horizontal cusp relation among 2-morphisms

deaths of spheres, and the attachments of 1-handles and 2-handles. Relations among the 3-morphisms would be generated by ambient isotopy of embedded 3-manifolds in 4-space. I expect that a complete list of relations can be given via the study of singularities between smooth 3-dimensional manifolds.

Conjecture. *The free monoidal* 3*-category with duals on one self dual (unframed) object generator is the* 3*-category of embedded* 3*-manifolds in* 4*-space.*

The 2-category of embedded surfaces. Given an embedding of a closed surface, F, in 3-space, it can be decomposed as a sequence of 2-morphisms as follows. A plane in 3-space that is disjoint from the surface F is chosen so that the projection of F to this plane has generic cusps and folds.This plane is called *the retinal plane.* General position considerations and Whitney's theorem guaranty that such a plane can be found; the set of planes form an open dense subset of the set of all planes. A height direction in the retinal plane is chosen so that the

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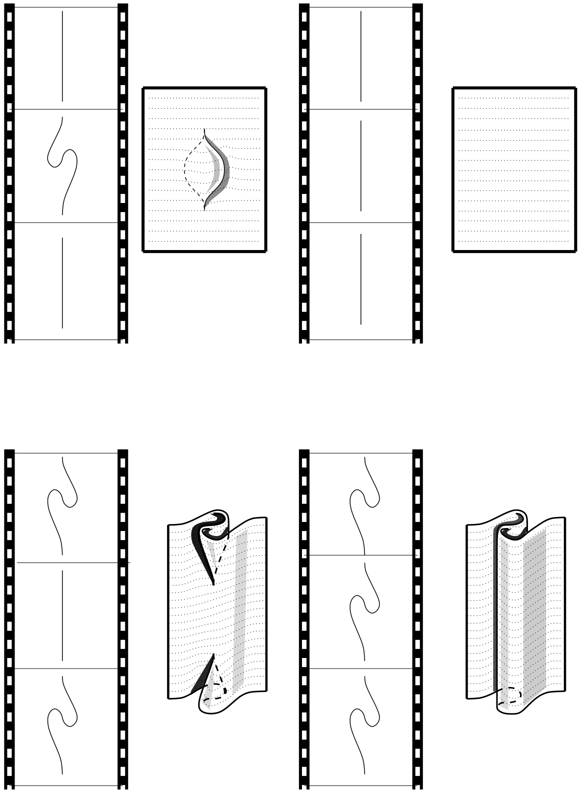

Figure 10: The lips and beak-to-beak relations among 2-morphisms

subsequent projection of the surface onto this line has non-degenerate critical points; that is, in local coordinates, the Hessian is non-singular. The projection of the surface onto the height axis can be given as the restriction of the projection of 3-space onto this axis. Choose a third direction perpendicular to the plane of projection: *a line of sight*, and a particular line parallel to this third direction. That is a line in 3-space whose direction vector is a scalar multiple of the vector defining the line of sight.

The intersection of the surface F with a general line parallel to the line of sight will consist of a finite collection of points such that the tangent to the surface at any such intersection maps injectively to the retinal plane. The distances between successive points on this line can be measured, and the set of these distances can be compared. In general, the intersections will not be integral multiples of a fixed length, but a subset of the positive integers can be established so that there is an order preserving map from the set of intersections of the given

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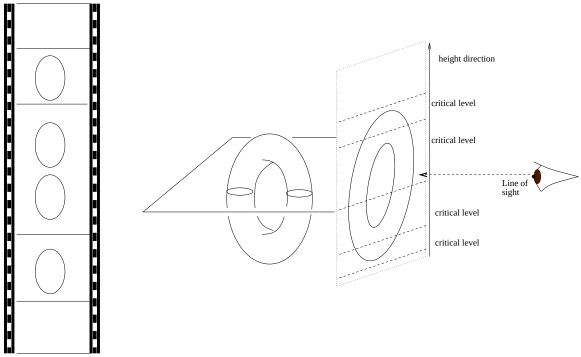

Figure 11: The retinal plane, height function, and resulting movie

line to a subset of integers that satisfies the following additional property: Points of intersection that are closer on the line of intersection correspond to similarly closer points on the subset of the integers. Thus not only is the order of intersection preserved but a new scale is chosen such that the relative distances are also preserved. The set of distances between points of F on the line of intersection can be partially ordered, and the subset of the integers is chosen so that if adjacent points of intersection x and y are closer together than the adjacent points z and w on the line of intersection, then the corresponding distances are closer ($|i_x - i_y| < |i_z - i_w|$ where $i_x, i_y, i_z, \&\ i_w$ are the corresponding integral points).

Any two lines parallel to the line of sight determine a plane, and the intersection of the given surface with this plane is a sequence of arcs. Critical points of the arcs, will project to points on the fold set in the retinal plane. If we consider one of the original lines as the bottom of an infinite rectangular strip, and the other line to be the top of such a strip, then the arcs undulate in this rectangular strip. This undulation is modeled via a sequence of generating 1-morphisms between the corresponding subsets of the positive integers.

As outlined in the previous two paragraphs, it is possible to start from the image of a given generic surface, measure carefully, and develop a sequence of 2-morphisms in the 2-category $\mathcal{EMB}_0$ that approximates the topography of the surface. To do so rigorously would require us to show that given surface can be suitably approximated. For example, for two lines of intersection at the same height level, results in two model subsets. The intersection of the surface with the thin strip that is bounded by these lines, consists of a sequence of arcs. These arcs have to be replaced by a sequence of 1-morphisms in $\mathcal{EMB}_0$, and the replacements have to be consistent from height to height. Specifically, folds and bulges have to be preserved. While I have not proven that there is a 2-isomorphism of 2-categories (a pair of 2-functors with a natural equivalences between their compositions and the identity functor), I feel confident that such

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a 2-equivalences can be constructed. My level of confidence is bolstered from my experience in drawing and shading surfaces. The drawings themselves are projections to the plane, and the technique that I use for depicting subtle details is exactly the idea of putting certain layers of the surface into a standard position, and isolating the singularities in specific locations.

Now in general, we can choose a sequence of parallel planes arranged as cosets of the height function. Literally the height function is defined on all of 3-space, and each plane is $\pi_t = h^{-1}(t)$ for some $t \in \mathbb{R}$. That is, for a given height, we choose a plane at that height, perpendicular to the height axis. For all but a finite number of heights, t, the intersection of $\pi_t = h^{-1}(t)$ with the surface, F, is generic and consists of a collection of closed loops. As such, it is modeled by the composition of 1-morphisms whose ultimate source and target are the empty sequence. A pair of nearby planes, between which are no critical or cuspal levels, will have intersection loops that differ only in position. On the other hand, two planes on either side of a critical point or cusp, will contain arcs with differing critical behavior, where now critical points are measured via a height function in this plane at a given vertical height.

The non-critical changes correspond to portions of the surface being closer or further from the retinal plane. It is these changes that we cease attempting to measure. The critical points corresponds to saddles, births, and deaths with respect to the height function, and the cusps correspond to changes in the fold set of the projection onto the retina.

Two embeddings of the surface that are ambiently isotopic can be related to one another via a sequence of relations among the 2-morphisms. In [18], we gave a graphical description of the ambient isotopy that converts a coffee cup into a doughnut. This isotopy involved the basic relations among folds that we have given here. It is amusing and satisfying to carefully watch the motions of a person while considering the changes in folds and cusps as the person moves. For example, the motion of the legs of a walking person as viewed from the side involves the commutation of distant fold lines. A knee bending and unbending can be approximated by a swallow-tail change. The junction of an arm and a shoulder involves a pair of cusps, the folds that are the profile of the biceps and triceps, and the fold of the arm pit. Line drawings, comics, and artist sketches exploit Whitney's theorem (or realize it) that states that the generic projection of one surface onto another is locally one-to-one and non-singular on all but a set of measure zero, and the measure zero set consists of fold lines that close or end in cusps.

2.2 Generic surfaces in 3-space

A closed surface in general position in 3-space has arcs of double points that end at branch points or triple points. In a neighborhood of a double point, the surface looks like the intersection of 2 coordinate planes in 3-space. In a neighborhood of a branch point the surface looks like the cone on a figure 8, and in a neighborhood

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of a triple point the surface resembles $\{(x, y, z) : xyz = 0\}$ — the set of three coordinate planes in 3-space.

We define the 2-category, $\mathcal{GEN}$, by introducing a new generating 1-morphism, X, that consist of the commutation of a pair of dots, and we introduce new 2-morphisms and relations among these. More specifically, we take objects to correspond to integers, $n = 0, 1, 2, \ldots$ as above. the object $\hat{n}$ is arranged as n dots along a line. Define an associative tensor product $\hat{n} \otimes \hat{m} = \widehat{n+m}$, and consider generating 1-morphisms $|$, $\cup$, $\cap$, and X. The diagrammatic depiction of X is given in Fig. 12.

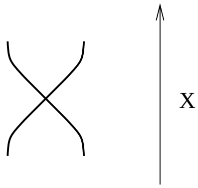

Figure 12: The generating 1-morphism X

The generating 2-morphisms consist of births and deaths of simple closed curves ($\emptyset \Rightarrow \cap \circ \cup$ or $\cap \circ \cup \Rightarrow \emptyset$), saddles ($\cup \circ \cap \Rightarrow ||$ and vice-versa), cusps ($(\cap \otimes |) \circ (| \otimes \cup) \Rightarrow |$ and vice-versa), branch points ($X \circ \cup \Rightarrow \cup$, vice-versa, $\cap \circ X \Rightarrow ||$ and vice-versa), cancelation of a pair of successive commutations ($X \circ X \Rightarrow ||$ and vice versa), the commutation of distant 2-morphisms, and a Yang-Baxter 2-morphism:

$$(X \otimes |) \circ (| \otimes X) \circ (X \otimes |) \Rightarrow (| \otimes X) \circ (X \otimes |) \circ (| \otimes X).$$

Representatives of each type of these 2-morphisms are given in Fig. 4. There is one additional 2-morphism to include. It is depicted separately in Fig. 13. (The reasons for separating this figure from Fig. 4 was forgetfulness rather than any mathematical reason. As I was finishing the manuscript, I realized I didn't have the source code for Fig. 4 at the place I was preparing the text). The 2-morphism can be described as $\psi : | \circ (\cap \otimes |) \circ (| \otimes X) \Rightarrow | \circ (| \otimes \cap) \circ (X \otimes |)$.

The 1-morphisms are akin to representatives of elements of the Brauer group, and the 2-morphisms are formed from relations therein, saddles, cusps, and births and deaths of simple closed curves.

A generic closed surface, F, in 3-space gives rise to a sequence of 2-morphisms from the empty 2-morphism and back as in the embedded surface case. The difference is the existence of the 1-morphisms X and the resulting 2-morphisms. These are branch points and triple points of the image of the surface.

From a categorical point of view, the 1-category that has objects $\{\hat{n}, n \in \{0, 1, 2, \ldots\}\}$ and morphisms generated by $|$, $\cup$, $\cap$, and X is a symmetric monoidal category. Upon imposing relations among the morphisms that correspond to some

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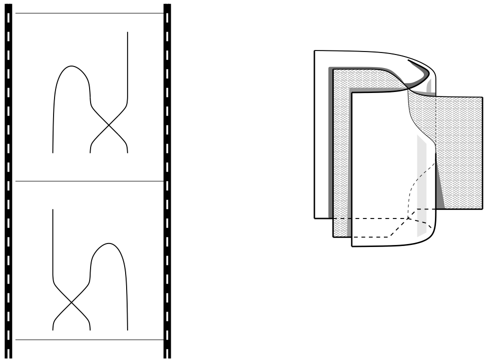

Figure 13: Moving a crossing over a fold

of the 2-morphisms (specifically the cusp relation gives rigidity and the branch point gives the category being pivotal), this symmetric category can be described algebraically. As an exercise, the reader should formulate a Freyd-Yetter type theorem that states that this category is the free symmetric, rigid, pivotal, monoidal category with duals on one self-dual object generator. The typical knot diagrammatic calculi can be used to perform computations in this category.

Homma-Nagase [31] and independently Roseman [44] demonstrated that there is a set of 7 local moves such that any two isotopic generic maps of closed surfaces in 3-dimensional manifolds can be transformed, one to the other, via a finite sequence of applications of these moves. In 3-space, any two orientable generic surfaces are isotopic if and only if the underlying surfaces are homeomorphic. Subsequently, Goryunov [24] gave a list of these moves as real pictures of codimension 1 singularities in the space of multi-germs of complex maps. The Roseman moves for generic surfaces have movie parameterizations. It is these movie parameterizations and movie moves that parameterize the interactions with the fold set (such as lips, beak-to-beak, swallowtails), and the interactions of the double point set, branch point set, and triple point set with the fold lines. See [18] for a full description. To understand these results in categorical language see [5].

At a category theory conference in the distant past, a mathematician raised strong objections to our use of diagrams to encode these categorical aspects. The defense of the diagrammatic point of view here is that the free symmetric monoidal 2-category on one self dual object generator is almost certainly the 2-category whose 1-morphisms are generated by $|$, X, $\cup$, and $\cap$, whose 2-morphisms are generated by the surfaces depicted in Fig. 4, and whose relations among 2-morphisms are given as projections of the movie moves. Here *projections* means that the classical knot crossings are projected to the pair of intersecting arcs depicted in Fig. 12. Therefore, a calculation in any symmetric monoidal 2-category,

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can be encoded as a surface manipulation. It has become standard to depict computations in braided monoidal categories in graphical notation. The advantage to me, is that the calculation can be followed by comparison of diagrams, and this comparison is easier for me, than the comparison of algebraic expressions. In the 2-category setting, we wind up manipulating surfaces. Most of the truly tedious algebraic work in my papers with Masahico and others is informed via diagrams. My colleague who objected to their use, obviously, had not seen the mathematics in ironing a shirt or shaping a metal plate via hammering.

The images of any two generic surfaces of the same genus are isotopic, and an isotopy can be constructed as an application of a finite sequence of moves taken from among the Roseman moves. If height function information is also preserved, then the moves are selected from among the projections of the movie moves given in [17] (see also the chart moves below). The categorical meaning of these results (modulo the proof that the category of generic surfaces is the free symmetric monoidal 2-category on one self-dual unframed object generator) is that invariants of generic surfaces defined categorically can, at best, detect genus.

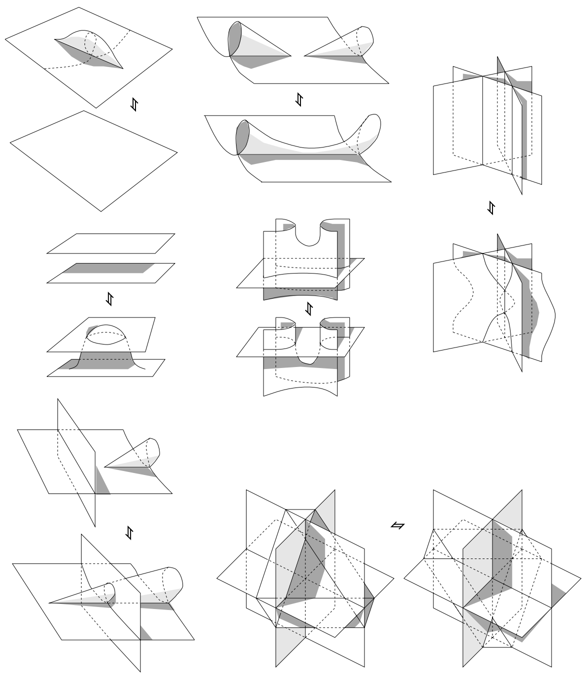

Figure 14: The Roseman moves

2.3 The 2-category of knotted surfaces in 4-space

The current section is completed by replacing the 1-morphism X — which is a transposition of adjacent elements in the permutation group — with a pair of 1-morphisms X and $\bar{X}$ which are positive and negative braid generators in the braid group, respectively. Having introduced over and under crossing information in the 1-morphisms, the crossing information is extended to the 2-morphisms. Let me describe this further.

The notion of a classical knot diagram is a planar picture that depicts crossing information. The diagram represents an observer's view of the knot; arcs at crossings that are further from the observer are depicted as haven been broken in the diagram. If the 3 space in which the knot lives is very thin in the observation direction, then most of the knot is on the plane of observation. The under crossing arcs bend behind that plane. Metaphorically, imagine a collection of wires mounted on a wall. In order to avoid shorting out a circuit, when wires might cross on the wall, the under crossing arc is fed behind the wall through a pair of small holes.

I am belaboring this point with knot diagrams because I want to generalize it to knotted surfaces. Now suppose that a surface is embedded in a 4-space that consists of $\mathbb{R}^3 \times (-3\epsilon, 0]$. Most of the surface is embedded in the 3-space $\mathbb{R}^3 \times \{0\}$ in which you are reading this article. However, there are small sections of the surface, along would-be double arcs, triple points and near branch points at which the surface bends below $\mathbb{R}^3 \times \{0\}$, and protrudes in to the "vinn" — a term coined by Rudy Rucker (Up is to down, as right is to left, as fore is to aft, as vinn is to vout). Along arcs of double points the surface has the structure of an interval times a semi-circle. At branch points this structure tapers off, and at the lower sheet at a triple point, there is a pair of canals that accommodates the canal at the middle sheet. Figures for these surfaces are given in [18].

The 2 category of 2-tangles is an algebraic model for knotted surfaces in 4-space. Here is the complete description. As before the objects are finite sets of points along a line. The 1-morphisms consist of $|$, $\cup$, $\cap$, X and $\bar{X}$. The 2-morphisms consists of cusps $C : (\cap \otimes |) \circ (| \otimes \cup) \Rightarrow |$ (and variants, births and deaths of simple closed curves, $B : \emptyset \Rightarrow (\cap \circ \cup)$ and $D : (\cap \circ \cup) \Rightarrow \emptyset$ respectively, and saddles $S : \cup \circ \cap \Rightarrow ||$. Each of these 2-morphisms has an analogue in which source and target 1-morphisms are switched. Finally, the lifts of the 2-morphisms depicted in Figs. 4 and 13 are included. Here *lifts* means that the 1-morphism X in these figures is replaced by the classical knot crossing, either X or $\bar{X}$, so that the resulting broken surfaces have consistent crossing information along their double curves. More specifically, the resulting 2-morphisms are the classical Reidemeister moves (including the ψ move), births, deaths, saddles, and cusps.

The relations among 2-morphisms can be encoded in terms of their effects on their projections to the retinal plane. In [17] we gave a set of chart moves that encrypted all of the movie moves in the resulting theory; the figures also appear in [18]. Again all of these can be described in purely categorical means as

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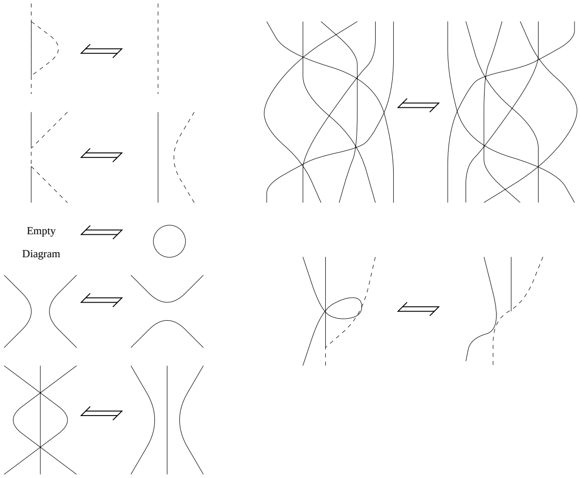

Figure 15: The chart moves page 1

in [5]. The following conventions were used in preparing these diagrams. A thin dashed line indicates a fold line in the retinal plane. A thick dashed line indicates either a fold or a crossing line. The crossings among thick dashed lines, then, are tensorators. A thin solid line represents a double arc from the projection of the knotted surface (originally in 4-space) to its diagram (in 3-space), and finally into the retinal plane. The three-fold intersection of solid lines represent triple points in the projection to 3-space (Reidemeister type III moves). The apparently tangential intersection between solid lines and dashed lines represents the 2-morphism ψ. The points at which arcs of double points appear to end at a fold line represent branch points (Reidemeister type I moves).

2.4 Why anyone else should care

An analogue of the Yang-Baxter equation is the Zamolodchikov equation (ZE) from statistical mechanics. In a braided monoidal 2-category, there is a solution to the ZE, and it has been shown [33], that a braided monoidal 2-category can be constructed from a solution to the ZE. Now our case of knotted surfaces has the added duality structure much of which I have hidden in the closet. Most of the duality has to do with the ability to include variants of the 2-morphisms obtained by reflecting them in various planes parallel to the faces of the boxes containing them. It has turned out that braided monoidal 2-categories with duals have been relatively difficult to find.

On the other hand some progress has been made. Most notably, is the Baez-Crans development [22, 3] in which categorifications of Lie Algebras give solutions

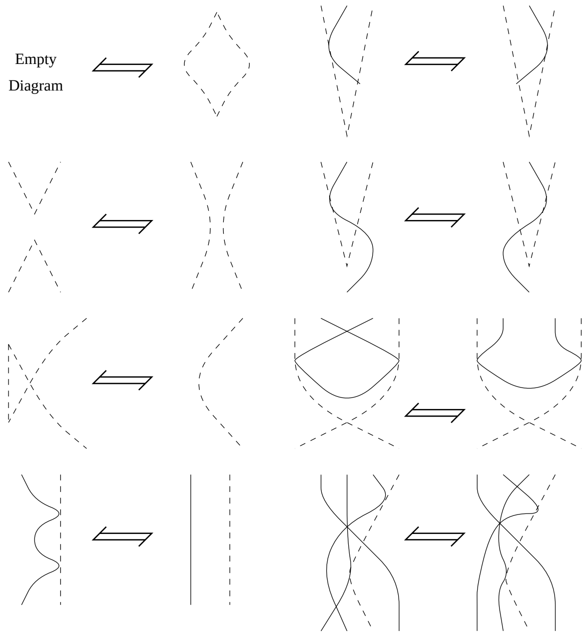

Figure 16: The chart moves page 2

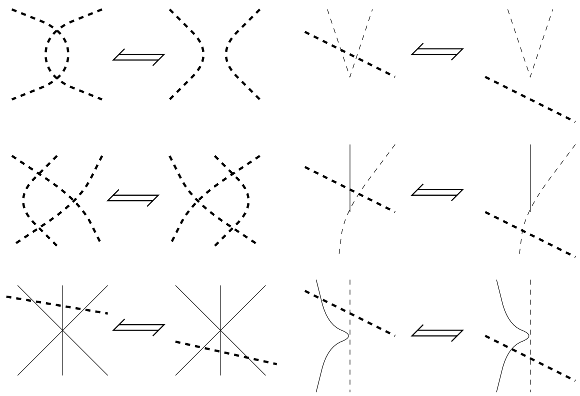

Figure 17: The chart moves page 3

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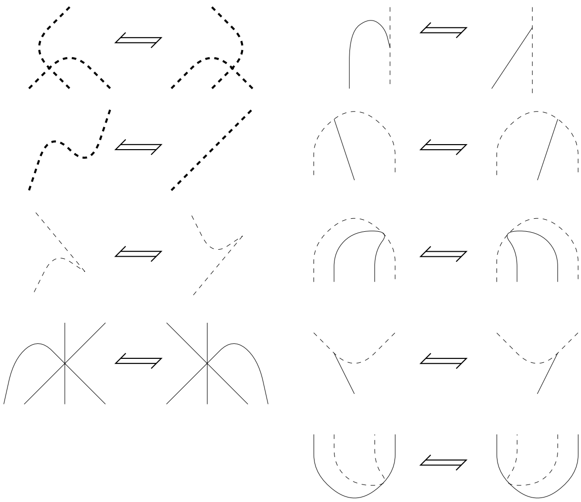

Figure 18: The chart moves page 4

to the ZE via the Jacobiator. Therein, they define an obstruction to the Jacobi identity holding, and from this obstruction, they solve the ZE.

Our own success in constructing invariants of knotted surfaces came from a careful look at Dijkgraaf-Witten invariants of 3-manifolds defined via finite groups. Therein group 3-cocycles were assigned to colored tetrahedra in a 3-manifold. The product over all tetrahedra was taken, and then the invariant sums over all colorings. There are normalization considerations as well. The point that I want to emphasize here is that the group cocycle condition is related to moves on triangulations. The ZE can be decomposed geometrically: This is the Roseman tetrahedron move. And so it makes sense to view the ZE as a cocycle condition. This was our approach to quandle cocycles.

3 Quandles

3.1 Quandles and knot colorings

A *quandle*, X, is a set with a binary operation $(a, b) \mapsto a * b$ such that

(I) For any $a \in X$, $a * a = a$.

(II) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.

(III) For any $a, b, c \in X$, we have $(a * b) * c = (a * c) * (b * c)$.

A *rack* is a set with a binary operation that satisfies (II) and (III). Racks and quandles have been studied in, for example, [9, 29, 32, 40].

The following are typical examples of quandles. A group G with conjugation

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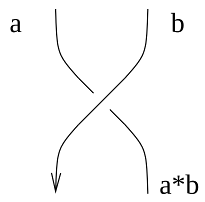

Figure 19: Quandle relation at a crossing

as the quandle operation: $a * b = bab^{-1}$, denoted by $X =$Conj(G), is a quandle. Any subset of G that is closed under such conjugation is also a quandle. Any $\Lambda(= \mathbb{Z}[t, t^{-1}])$-module M is a quandle with $a * b = ta + (1-t)b$, $a, b \in M$, that is called an *Alexander quandle*. Let n be a positive integer, and for elements $i, j \in \{0, 1, \dots, n-1\}$, define $i * j \equiv 2j - i \pmod{n}$. Then $*$ defines a quandle structure called the *dihedral quandle*, R_n.

Let X be a fixed quandle. Let K be a given oriented classical knot or link diagram, and let $\mathcal{R}$ be the set of (over-)arcs. The normals (normal vectors) are given in such a way that the ordered pair (tangent, normal) agrees with the orientation of the plane, see Fig. 19. A (quandle) *coloring* $\mathcal{C}$ is a map $\mathcal{C} : \mathcal{R} \to X$ such that at every crossing, the relation depicted in Fig. 19 holds. The (ordered) colors $\mathcal{C}(\alpha), \mathcal{C}(\beta)$ are called *source* colors. Let $\mathrm{Col}_X(K)$ denote the set of colorings of a knot diagram K by a quandle X.

The cocycle invariant for classical knots [15] was defined as follows. Let $\phi \in C^2_{\mathrm{Q}}(X; A)$ be a 2-cocycle of a finite quandle X with the coefficient group A. This ϕ is regarded as a function $X \times X \to A$ that satisfies the 2-cocycle condition

$$\phi(x, y) - \phi(x, z) + \phi(x * y, z) - \phi(x * z, y * z) = 0, \quad \forall x, y, z \in X$$

and $\phi(x, x) = 0, \forall x \in X$. Let $\mathcal{C}$ be a coloring of a given knot diagram K by X. The *Boltzmann weight* $B(\mathcal{C}, \tau)$ at a crossing τ of K is then defined by $B(\mathcal{C}, \tau) = \phi(x_\tau, y_\tau)^{\epsilon(\tau)}$, where x_τ, y_τ are source colors at τ and $\epsilon(\tau)$ is the sign (± 1) of τ. In Fig. 19, it is a positive crossing if the under-arc is oriented downward. Here $B(\mathcal{C}, \tau)$ is an element of A written multiplicatively. The formal sum (called a state-sum) in the group ring $\mathbb{Z}[A]$

$$\Phi_\phi(K) = \sum_{\mathcal{C} \in \mathrm{Col_X(K)}} \prod_\tau B(\mathcal{C}, \tau)$$

is called the quandle cocycle invariant.

Theorem 3.1 [15] *The state-sum $\Phi_\kappa(K)$ does not depend on the choice of a diagram D of a given knot K, so that it is a well-defined knot invariant.*

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The cocycle invariant can be also written as a family (multi-set, a set with repetition allowed) of weight sums [39]

$$\left\{ \sum_{\tau} B(\mathcal{C}, \tau) \;\middle|\; \mathcal{C} \in \mathrm{Col}_{\mathrm{X}}(\mathrm{K}) \right\}$$

where now the values of B in A are denoted by additive notation.

Generalizations have been discovered [11, 12, 13]. The quandle cocycle invariants have also been defined for knotted surfaces in 4-space, in a similar manner, using quandle 3-cocycles.

Among my current goals with Masahico Saito is to develop a 2-categorical notion of quandles that includes quandle 3-cocycles as giving 2-morphisms. This goal was established in the early versions of the current paper, but it has not yet come to fruition. Stay tuned for further information.

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