Quandles, (co)-homology, applications, and beyond

J. Scott Carter

University of South Alabama

January 2010
7th East Asia School of Knots
Hiroshima, Japan
Outline

1. Thanks to organizers
2. Thanks to collaborators
3. Recent contributions of a number of authors
4. Introduce quandles
5. Examples
6. Groups associated to a quandle
7. Homology of self-distributive binary operations
8. $G$-families of quandles
9. unifying quandle and group cohomology
The ABCs of quandles

a Automorphic set, Andruskiewitsch, Asami
b Brieskorn, Baez
c JSC, Clauwens, Conway, Crans
d Distributive groupoid
e Eisermann, Elhamdadi, Etingof
f Fenn
g Graña
h Hatakenaka
i Inoue, Ishii, Iwakiri
j Jacobsson, Jang, Jelsovsky, Joyce
k Kabaya, Kamada, Kimura, Kishimoto
l Langford, Litherland, Lopes
m Matveev, Mochizuki
n Nakamura, Nelson, Niebrzydowski, Nosaka
o Oshiro
p Przytycki
q Quandle
r Rourke, Rubensztein
s Saito, Satoh, Shima, Sanderson
t Takasaki, Tanaka
v Vo, Vassily Manturov
w Wraith
y Yoshiro, Yun
z Zablow
Recent contributions (alphabetically)

Clauwens

- proves the delayed Fibonacci conjecture of Niebrzydowski-Przytycki
- gives a concise description of the associated group of a quandle
- lists all connected simple quandles of order $\leq 14$

Elhamdadi and students describe the autom. group of several small order quandles.
Hatakenaka

- gives lower bounds for the triple point number of the 2-twist-spun $T(5,2)$
- relates the quandle cocycle invariant to Dijkgraaf-Witten invariant
- with Nosaka relates four-fold symmetric quandle homotopy invariants to Dijkgraaf-Witten invariants.
Inoue and Inoue/Kabaya

- interpret Hyperbolic volume and Chern Simons as quandle cocycle invariants (parabolic elements of $PSL(2, \mathbb{C})$ under conjugation)
- define a simplicial quandle homology and interrelate this to group homology
- Kabaya shows how to reconstruct Mochizuki’s 3-cocycle from a group cocycle and relates this to cyclic branched covers.

¿Can these techniques be generalized to re-prove Nosaka/Clauwens result?
Ishii and Iwakiri modify quandle cohomology to define a shadow colored invariant for knotted spacial graphs and knotted handle-bodies.

Ishii, Iwakiri, Jang, and Oshiro are further developing these ideas in a notion of $G$-family of quandles.
Kimura characterized framings of classical knots via quandle homology classes. This relies on work of Eisermann that shows that $K$ is knotted iff $H^Q_2(\pi_Q(K)) \cong \mathbb{Z}$, and Litherland and Nelson’s splitting of quandle homology. Kimura shows that a framing corresponds to a unique quandle cocycle.

Nelson’s recent work focuses upon bi-racks and bi-keis.
Nosaka

- gave a proof of the delayed Fibonacci conjecture and generalized beyond dihedral quandles
- studies and relates quandle cocycle invariants to quandle homotopy groups
- detects a serious error in the tables of computations of CEGS
Breathe
A quandle \((X, \triangleleft)\) is a set \(X\) equipped with a binary operation

\[
\triangleleft : X \times X \to X
\]

that satisfies:

(i) \(x \triangleleft x = x\) for all \(x \in X\)

(ii) For all \(x, y \in X\), there exists a unique \(z \in X\) such that \(z \triangleleft x = y\)

\[z = x \triangleleft^{-1} y\]

(iii) \((x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)\) for all \(x, y, z \in X\)
\[ \pi_Q(K) \]

The fund. quandle of a codim. 2 embedding:

Base point

\[ x_0 \quad x_1 \]

\[ \partial N(K) \]

\[ x_0 \quad x_1 \]

\[ \partial N(K) \]

\[ x \quad y \]

\[ x < y \]
\( \pi_1(K) \) acts upon \( \pi_Q(K) \)

The fund. group acts upon the fund. quandle via
path multiplication:
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]

\[ K \]
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path
The Peripheral subgroup

The peripheral subgroup stabilizes the constant path

\[ \partial N(K) \]
\[ \pi_Q(K) \text{ classifies knots} \]

There is a quandle str. on the coset space \( \pi_1(K)/P \) where \( P \) is the periph. subgrp. Let \( \mu_* \) denote the meridonal class at the base point, with \( * \in \partial N(K) \). Define \( P\gamma \lhd P\delta = P\gamma\delta^{-1}\mu_*\delta \).

**Theorem**

*(Joyce Matveev)* There is a quandle isom. between the quandles \((\pi_1(K), P, \lhd)\) and \(\pi_Q(K)\).

**Corollary**

\(\pi_Q(K)\) classifies \( K \) up to ori. rev. homeom.
Breathe
Examples of quandles 1

0. Any set $X$ with binary operation $a \triangleleft_0 b = a$ is called the trivial quandle.

1. Let $H$ be a group $G = \text{Aut}(H)$, $K \subset G$ a subgrp, and $s \in G$ an element that fixes each $k \in K$: i.e. $k = (k)s$. On $G/K$ define $Ka \triangleleft_s Kb = K(ab^{-1})s b$.

Remark. (Joyce/Matveev) Every quandle can be obtained as a variation on example 1.
Examples of quandles 2

3. $H$ is an abelian group, $T : H \to H$ is a right autom. $a \triangleleft b = aT + b - bT$. This is an Alexander quandle.

4. $S \subset G$ a subset that is closed under conjugation. $a \triangleleft b = b^{-1}ab$.

5. Specifically, if $S = G$, then Conj : $\mathcal{G} \to \mathcal{Q}$ is a functor from the cat. of groups to the cat. of quandles.

6. Core : $\mathcal{G} \to \mathcal{Q}$ is another functor from the cat. of groups to the cat. of quandles: $a \triangleleft b = ba^{-1}b$. 
Breathe
Groups that arise from quandles

Let $X$ denote a quandle. Its autom. grp. is

$$\text{Aut}(X) = \{ \phi : X \to X : (x \triangleleft y)\phi = (x)\phi \triangleleft (y)\phi \quad \& \quad \phi \text{ is biject.} \}.$$ 

Quandle axiom, ii, and iii imply that $X$ acts upon itself as the group of inner autom. Observe, $\exists F(X) \to \text{Inn}(X)$ a surj. from the free grp. gen. by $X$ to the inner autom. grp.

$$\text{As}(X) = \text{AdConj}(X) = \langle |X| : a \triangleleft b = b^{-1}ab \rangle$$

$$\text{AdCore}(X) = \langle |X| : a \triangleleft b = ba^{-1}b \rangle$$
Breathe
Quandle homology

(Przytycki’s approach) Suppose $X$ is a set. Let $\text{Bin}(X) = X^{X \times X}$ denote the set of bin. oper. on $X$. $\text{Bin}(X)$ is a monoid: $a(\triangleleft_1 \cdot \triangleleft_2)b = (a \triangleleft_1 b) \triangleleft_2 b$. The trivial quandle $a \triangleleft_0 b = a$ serves as the identity.

Given $\triangleleft \in \text{Bin}(X)$ define

$$\partial_n^{\triangleleft}(x_1, \ldots, x_n)$$

$$= \sum_{i=2}^{n} (-1)^i (x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n)$$

on the free ab. grp. gen by $n$-tuples of elems. from $X$. 
homology cont.

\[ \partial_n^<(x_1, \ldots, x_n) = \sum_{i=2}^{n} (-1)^i (x_1 \triangleleft x_i, \ldots, x_{i-1} \triangleleft x_i, x_{i+1}, \ldots, x_n) \]

is a differential \textit{i.e.} \( \partial_{n-1} \circ \partial_n = 0 \) iff \( \triangleleft \) is self-distr: \((x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)\). More generally,

\[(\partial_{n-1}^< - \partial_{n-1}^<) \circ (\partial_n^< - \partial_n^<) = 0\]

iff \((x \triangleleft_i y) \triangleleft_j z = (x \triangleleft_j z) \triangleleft_i (y \triangleleft_j z) \ \forall i, j.\)
Observe that the trivial quandle is self-distr. and distributes over any other bin. op. The differentials in rack homology are defined as $(\partial_n^{-} - \partial_n^{<0})$. Quandle homology is defined by neglecting chains of the form $(x_1, \ldots, x_n)$ where $x_i = x_{i+1}$ for some $i = 1, \ldots, n - 1$. 
Quandle cocycles

Quandle 2 and 3-cocycles are functions $\phi$, and $\theta$ with values in an ab. grp. s.t. $\phi(x, x) = 0$, $\theta(x, x, y) = \theta(x, y, y) = 0$ and the conditions depicted below hold.

\[
\phi(x, y) + \phi(xy, z) + \phi(y, z) = \phi(y, z) + \phi(x, z) + \phi(xyz, yz)
\]

\[
\theta(w, x, y) + \theta(wy, xyz, z) + \theta(wy, z) = \theta(w, x, z) + \theta(w, y, z) + \theta(wz, y, x)
\]
... after Ishii-Iwakiri-Jang-Oshiro. $G$ is a group. $X$ is a set. $\forall g \in G \ \exists$ quan. str. $\triangleleft_g$ on $X$ s.t.

- $(a \triangleleft_g b) \triangleleft_h b = a \triangleleft_{gh} b$
- $\triangleleft_{g^{-1}} = (\triangleleft_g)^{-1}$
- $(a \triangleleft_g b) \triangleleft_h c = (a \triangleleft_h c) \triangleleft_{h^{-1} gh} (b \triangleleft_h c)$.
Examples

1. Let $H$ be a group. $G = \text{Aut}(H)$. Define $a \triangleleft_s b = s(ab^{-1})b$.

2. Specifically, $X = (\mathbb{Z}/(p))^n$ — row vectors $G = \text{SL}(n, \mathbb{Z}/(p))$, and $a \triangleleft_M b = aM + b - bM$. 
Lemma

Let \((X, G)\) denote a \(G\) family of quandles on \(X\). Then \(G \times X\) is a quandle under the binary operation \((g, a) \triangleleft (h, b) = (h^{-1}gh, a \triangleleft_h b)\).

\(G \times X\) is a dynamical extension of \(\text{Conj}(G)\) with the dyn. cocycle \(\alpha : G \times G \to X^{X \times X}\) given by \(\alpha_{g,h}(x, y) = a \triangleleft_h b\). In particular,

\[
\alpha_{g \triangleleft_h k}(\alpha_{g,h}(a, b), c) = \alpha_{g \triangleleft_h k, h \triangleleft_k}(\alpha_{g,k}(a, c), \alpha_{h,k}(b, c)).
\]
Ishii-Iwakiri-Jang-Oshiro are using this construction to give invariants of spacial graphs and knotted handle-bodies. To finish this talk, I want to modify their idea to present a cohomology theory that combines group and quandle cohomology.
2 cochains

\[ \eta((g,a),(h,a)) \quad \phi((g,a),(h,b)) \]
2 cocycle conditions

\[ (hk)g(hk)^{-1}hgh^{-1}k^{-1}hgh^{-1}k^{-1}kghk^{-1}kgh^{-1}khk^{-1}kgh^{-1}kgh^{-1}g \]

\[ (hk)g(hk)^{-1}hgh^{-1}k^{-1}hgh^{-1}k^{-1}kghk^{-1}kgh^{-1}khk^{-1}kgh^{-1}kgh^{-1}g \]
2 cocycle conditions

\[ \phi((g, a), (h, b)) + \phi((g \triangleleft h, a \triangleleft_h b), (c, k)) = \]
\[ \phi((g, a), (k, c)) + \phi((g \triangleleft k, a \triangleleft_k c), (h \triangleleft k, b \triangleleft_k c)); \]

\[ \eta((g, a), (h, a)) + \eta((gh, a), (k, a)) = \]
\[ \eta((h, a), (k, a)) + \eta((g, a), (hk, a)); \]

\[ \eta((g, a), (h, a)) + \phi((gh, a), (k, c)) = \]
\[ \phi((h, a), (k, c)) + \phi((g, a), (k, c)) \]
\[ + \eta((g \triangleleft k, a \triangleleft_k c), (g \triangleleft h, a \triangleleft_c c)). \]

\[ \eta((h, b), (k, b)) + \phi((g, a), (hk, b)) = \]
\[ \phi((g, a), (h, b)) + \phi((g \triangleleft h, a \triangleleft_h b), (k, b)) \]
\[ + \eta((h, b), (k, b)). \]
3 cocycle conditions, part 1
1st 3-cocycle cond.

\[ \alpha((g, a), (h, a), (k, a))) + \alpha((g, a), (hk, a), (\ell, a)) \\
+ \alpha((h, a), (k, a), (\ell, a)) \\
= \alpha((gh, a), (k, a), (\ell, a)) + \alpha((g, a), (h, a), (k\ell, a)) \]
3 cocycle conditions, part 2
2nd 3-cocycle cond.

\[
\psi_1((g, a), (h, a); (k, b)) \\
+ \psi_1((g \triangleleft k, a \triangleleft_k b), (h \triangleleft k, a \triangleleft_k b); (\ell, c)) \\
+ \theta((g, a), (k, b), (\ell, c)) + \theta((h, a), (k, b), (\ell, c)) \\
= \theta((gh, a), (k, b), (\ell, c)) + \psi_1((g, a), (h, a); (k, c)) \\
+ \psi_1((g \triangleleft \ell, a \triangleleft_\ell c), (h \triangleleft \ell, a \triangleleft_\ell c); (k \triangleleft \ell, b \triangleleft_\ell c))
\]
3 cocycle conditions, part 3
The 3rd 3-cocycle condition is given by:

\[
\psi_2((g, a); (h, b), (k, b)) + \psi_1((h, b), (k, b); (\ell, c))
+ \theta((g \triangleleft h, a \triangleleft_h b), (k, b), (\ell, c))
+ \theta((g, a), (h, b), (\ell, c))
\]

\[
= \theta((g, a), (hk, b), (\ell, c)) + \psi_1((h, b), (k, b); (\ell, c))
+ \psi_2((g \triangleleft \ell, a \triangleleft_\ell c); (h \triangleleft \ell, b \triangleleft_\ell c), (k \triangleleft \ell, b \triangleleft_\ell c))
\]
3 cocycle conditions, part 4
4th 3-cocycle cond.

\[
\psi_2((h, b); (k, c), (\ell, c)) + \psi_2((g, a); (k, c), (\ell, c)) \\
- \theta((g \triangleleft k, a \triangleleft_k c), (h \triangleleft k, b \triangleleft_k c), (\ell, c)) \\
= -\theta((g, a), (h, b), (k\ell, c)) \\
+ \psi_2((g \triangleleft h, a \triangleleft_h b); (k, c), (\ell, c)) \\
+ \psi_2((h, b); (k, c); (\ell, c)) \theta((g, a), (h, b), (k, c))
\]
3 cocycle conditions, part 5
5th 3-cocycle cond.

\[ \psi_1((g,a),(h,a);(k\ell,b)) + \psi_2((h,a);(k,b),(\ell,b)) + \psi_2((g,a);(k,b),(\ell,b)) = \psi_2((gh,a);(k,b),(\ell,b)) + \psi_1((g,a),(h,a);(k,b)) + \psi_1((g \triangleleft k,a \triangleleft_k b),(h \triangleleft k,a \triangleleft_k b);(\ell,b)) \]
3 cocycle conditions, part 6
6th 3-cocycle cond.

\[
\psi_1((gh, a), (k, a); (\ell, b)) + \psi_1((g, a), (h, a); (\ell, b)) \\
+ \alpha((g \triangleleft \ell, a \triangleleft \ell b), (h \triangleleft \ell, a \triangleleft \ell b), (k \triangleleft \ell))
\]

\[
= \alpha((g, a), (h, a), (k, a)) + \psi_1((g, a), (hk, a); (\ell, b)) \\
+ \psi_1((h, a), (k, a); (\ell, b))
\]
3 cocycle conditions, part 7
7th 3-cocycle cond.

\[
\begin{align*}
\theta((g, a), (h, b), (k, c)) &+ \theta((g \triangleleft k, a \triangleleft_k c), (h \triangleleft k, b \triangleleft_k c), (\ell, d)) \\
&+ \theta((g, a), (k, c), (\ell, d)) \\
= \theta((g \triangleleft h, a \triangleleft_h b), (k, c), (\ell, d)) &+ \theta((g, a), (h, b), (\ell, d)) \\
&+ \theta((g \triangleleft \ell, a \triangleleft_\ell d), (h \triangleleft \ell, b \triangleleft_\ell d), (k \triangleleft \ell, c \triangleleft_\ell d)
\end{align*}
\]
Conclusion

In general, the $n$-cocycles conditions are given by products such as $n$-dimensional solids $\Delta_i \times \Box_j$ and $\Box_j \times \Delta_i$ where $\Delta_i$ is an $i$-dimensional simplex, $\Box_j$ is a $j$-dimensional cube, and $i + j = n$. In general, we consider products $\Delta \times \Box \times \Delta \cdots$ and take the ordinary geometric boundary while keeping track of crossing information.
We anticipate interesting invariants for 3-valent graphs and embedded foams.
Domo Arigato

Thank you for your attention!
Thanks to the organizers for inviting me here!