In this handout, we are going to learn how to perform elementary row operations on matrices, via performing the corresponding operation on the appropriate sized identity matrix. The row operations are exactly what are needed in order to take an arbitrary system of equations, and solve the system.

The goal will be to put the matrix in either row echelon form or reduced row echelon form. This technique is also known as Gaussian elimination. Essentially, we take one equation, solve for one variable in terms of the others, and back substitute that equation into the previous equation. The process continues for as long as possible. At the end of the process we determine one of the three possibilities:

1. that the system of equations is inconsistent (for example when two of the equations represent parallel hyper planes) and there is no solution;

2. that the system of equations has an infinite number of solutions (for example this happens when we have one linear equation in several unknowns); or

3. the system of equations has a unique solution. In this case, there are necessarily \( \textit{at least} \) as many equations as there are unknowns. Having the same number of equations as unknowns is not sufficient for the solution to be unique.

Case (2) and (3) are called \textit{consistent systems}.

In general, given a system of equations we may perform the following three operations without affecting the solution set.

1. Interchange two equations.

2. Multiply both sides of a given equation through by a non-zero constant.

3. Add or subtract a given equation from the others.
Consider a generic system of $m$ equations in $n$ unknowns \((x^1, x^2, \ldots, x^n)\) **superscripts**

**NOT exponents:**

\[
\begin{align*}
  a_1^1 x^1 &+ a_2^1 x^2 + \ldots + a_n^1 x^n = b^1 \\
  a_1^2 x^1 &+ a_2^2 x^2 + \ldots + a_n^2 x^n = b^2 \\
  \vdots & \quad \vdots \quad \ldots \quad \vdots \\
  a_1^m x^1 &+ a_2^m x^2 + \ldots + a_n^m x^n = b^m 
\end{align*}
\]

There is an obvious associated (augmented) matrix:

\[
\begin{bmatrix}
  a_1^1 & a_2^1 & \ldots & a_n^1 & | & b^1 \\
  a_1^2 & a_2^2 & \ldots & a_n^2 & | & b^2 \\
  \vdots & \vdots & \ldots & \vdots & | & \vdots \\
  a_1^m & a_2^m & \ldots & a_n^m & | & b^m
\end{bmatrix}
\]

The three operations to the equations listed above correspond to three operations on the augmented matrix:

1. Interchange any two rows.
2. Multiply a given row by a non-zero constant.
3. Add or subtract a given row from another row.

Observe that items 2 and 3 may by combined to
4. Add a non-zero multiple of one row to another row.

**Example.** Consider the system:

\[
\begin{align*}
  x &+ y + z = 1 \\
  2x &+ z = 4 \\
  y &+ z = 2
\end{align*}
\]

and the associated augmented matrix:

\[
\begin{bmatrix}
  1 & 1 & 1 & 1 \\
  2 & 0 & 1 & 4 \\
  0 & 1 & 1 & 2
\end{bmatrix}
\]

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By subtracting the 3rd row from the 1st \((R_1 - R_3 \mapsto R_1)\), we obtain:
\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2 \\
\end{bmatrix}.
\]

By interchanging the 2nd row and the 3rd row \((R_2 \leftrightarrow R_3)\), we obtain
\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 2 \\
2 & 0 & 1 & 4 \\
\end{bmatrix}.
\]

Next subtract 2 times the 1st row from the 3rd row \((R_3 - 2R_1 \mapsto R_3)\), to obtain
\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 2 \\
0 & 0 & 1 & 6 \\
\end{bmatrix}.
\]

At this point, we see that \(z = 6\) and we can back substitute to complete the solution. The resulting matrix is in row echelon form: in the sense that the leading non-zero entries in the rows are all 1s and these proceed from left to right as you move down the matrix. The row echelon form is not unique though. Meanwhile the reduced row echelon form is unique, so we will perform one more step. Subtract the 3rd row from the 2nd \((R_2 - R_3 \mapsto R_2)\), to obtain
\[
\begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 6 \\
\end{bmatrix}.
\]

The solution is \(x = -1\), \(y = -4\), and \(z = 6\). These three planes intersect at the point
\[
\begin{bmatrix}
-1 \\
-4 \\
6 \\
\end{bmatrix}.
\]

Next, I want to show you how to obtain the same result by applying the same sequence of operations to the \((3 \times 3)\) identity matrix, and gradually multiplying by the result.

In general, the \((n \times n)\) identity matrix is the square matrix that has 1s on the diagonal and 0s elsewhere. It is denoted \(I\) or \(I(n)\).

\[
I(n) = \begin{bmatrix}
1 & 0 & \ldots & 0 \\
0 & 1 & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & 1 \\
\end{bmatrix}
\]
In the \((3 \times 3)\)-case,

\[
I(3) = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Now, subtract the 3rd row from the 1st \((R_1 - R_3 \rightarrow R_1)\), to obtain:

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
\]

Multiply this on the left of the given matrix:

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix}.
\]

Next to the matrix

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

interchange the 2nd row and the 3rd row \((R_2 \leftrightarrow R_3)\), to obtain

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}.
\]

We apply \textit{this result to the original augmented matrix}

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix} \cdot \begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix}.
\]

Also observe that

\[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix},
\]

and

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix} \cdot \begin{bmatrix}
1 & 0 & 0 & -1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix} = \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 1 & 2 \\
2 & 0 & 1 & 4
\end{bmatrix}.
\]
Apply the row operation \((R_3 - 2R_1 \mapsto R_3)\), to the result of the previous row operations on the identity:

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
-2 & 1 & 2
\end{bmatrix}.
\]

Apply the operation \((R_2 - R_3 \mapsto R_2)\),

\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
-2 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & -1 \\
2 & -1 & -1 \\
-2 & 1 & 2
\end{bmatrix}.
\]

Finally, apply this result to the original augmented matrix:

\[
\begin{bmatrix}
1 & 0 & \ -1 \\
2 & -1 & -1 \\
-2 & 1 & 2
\end{bmatrix}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & -1 \\
0 & 1 & 0 & -4 \\
0 & 0 & 1 & 6
\end{bmatrix}.
\]

A sequence of elementary row operations can be used to put any augmented system into reduced row echelon form. Applying the same operations to the identity matrix in turn, and multiplying the result on the left by the next operation results in a matrix that achieves the same result when multiplied on the left of the augmented system.

For each of the following systems, perform the corresponding elementary operations, both to the associated augmented matrices, and to the identity matrix. Observe how the system of equations is transformed.

1. \[
3x + 4y = 24 \\
2x + y = 0
\]

2. \[
2x - 5y = 30 \\
x + y = 10
\]

3. \[
x + y = 1 \\
2x - y = 2
\]

4. \[
5x - 4y = 20 \\
3x + 2y = 6
\]
5. 
\[ 3x + 4y = 24 \]
\[ x - 2y = 8 \]

6. 
\[ 2x - 5y = 30 \]
\[ x - y = 0 \]

7. 
\[ 3x + 4y + 8z = 24 \]
\[ x = 0 \]
\[ y = 0 \]

8. 
\[ 2x - 5y - 6z = 30 \]
\[ x = 0 \]
\[ z = 0 \]

9. 
\[ x + y + z = 1 \]
\[ x = 0 \]
\[ z = 0 \]

10. 
\[ 5x + 4y - 2z = 20 \]
\[ x = 0 \]
\[ z = 0 \]

11. 
\[ 3x + 4y + 8z = 24 \]
\[ y = 0 \]
\[ z = 0 \]

12. 
\[ 3x + 4y + 8z = 24 \]
\[ x + y = 2 \]
\[ 2x + y + z = 4 \]

13. 
\[ 2x - 5y - 6z = 30 \]
\[ x - y + z = 1 \]
\[ x + y - z = 1 \]
14. 
\begin{align*}
x + y + z &= 1 \\
x + z &= 2
\end{align*}

15. 
\begin{align*}
5x + 4y - 2z &= 20 \\
x + 2y + 2z &= 5
\end{align*}