In general, the matrix product is defined for a pair of matrices $A \in M(n, k)$ and $B \in M(k, m)$. The notation indicates that $A$ is a matrix with $n$ rows and $k$ columns while $B$ is a matrix with $k$ rows and $m$ columns. That is, the number of columns of $A$ is equal to the number of rows of $B$.

The first example that we have experienced is the dot product. For example, if

$$A = [a_1, a_2, \ldots, a_k]$$

and

$$B = \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^k \end{bmatrix},$$

then

$$A \cdot B = [a_1, a_2, \ldots, a_k] \cdot \begin{bmatrix} b^1 \\ b^2 \\ \vdots \\ b^k \end{bmatrix} = a_1 b^1 + a_2 b^2 + \cdots + a_k b^k = \sum_{\ell=1}^{k} a_\ell b^\ell.$$
In this preliminary numeric case, $A$ is a $(1 \times 3)$-matrix, and $B$ is a $(3 \times 1)$-matrix.

In general if $A$ is an $(n \times k)$-matrix and $B$ is a $(k \times m)$-matrix, then the matrix product $A \cdot B$ is an $(n \times m)$-matrix in which the $(i,j)$th entry is the dot product of the $i$th row of $A$ with the $j$th column of $B$. If $a_{i}^{\ell}$ is the entry in the $i$th row and $\ell$th column of the matrix $A$ and $b_{\ell}^{j}$ is the entry in the $\ell$th row and $j$th column of $B$, then the matrix entry in the $i$th row, $j$th column of the product is $\sum_{\ell=1}^{k} a_{i}^{\ell} b_{\ell}^{j}$. Usually, when we first see the formula, it scares us, so let’s see an example.

Let $A = \begin{bmatrix} 3 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix}$ and let $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$. Then

$A \cdot B = \begin{bmatrix} 3 & 1 & 2 \\ -4 & 2 & 5 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{bmatrix}$

$= \begin{bmatrix} 3 \cdot 1 + 1 \cdot 2 + 2 \cdot 3 & 3 \cdot 2 + 1 \cdot 4 + 2 \cdot 6 & 3 \cdot 3 + 1 \cdot 6 + 2 \cdot 9 \\ -4 \cdot 1 + 2 \cdot 2 + 5 \cdot 3 & -4 \cdot 2 + 2 \cdot 4 + 5 \cdot 6 & -4 \cdot 3 + 2 \cdot 6 + 5 \cdot 9 \end{bmatrix} = \begin{bmatrix} 11 & 22 & 33 \\ 15 & 30 & 45 \end{bmatrix}$.

Compute the matrix product of the following matrices.

1. $\begin{bmatrix} 1/3 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -4 \\ 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} 1/3 & -4/3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 3 & 4 & 24 \\ 0 & 1 & 0 \end{bmatrix}$

3. $\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 \\ 0 & 1 \end{bmatrix}$

4. $\begin{bmatrix} 1 & -5/2 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & -5 & 30 \\ 0 & 1 & 0 \end{bmatrix}$

5. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

6. $\begin{bmatrix} 1/5 & 0 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 5 & -4 \\ 0 & 1 \end{bmatrix}$
7. \[
\begin{bmatrix}
1 & -4/5 \\
0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
5 & -4 & 20 \\
0 & 1 & 0
\end{bmatrix}
\]

8. \[
\begin{bmatrix}
1 & 0 \\
-3 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 1 \\
1 & 0
\end{bmatrix}
\]

9. \[
\begin{bmatrix}
1 & 0 \\
0 & 1/4
\end{bmatrix}
\cdot
\begin{bmatrix}
0 & 1 \\
1 & -3
\end{bmatrix}
\]

10. \[
\begin{bmatrix}
0 & 1 \\
1/4 & -3/4
\end{bmatrix}
\cdot
\begin{bmatrix}
3 & 4 & 24 \\
1 & 0 & 0
\end{bmatrix}
\]

11. \[
\begin{bmatrix}
1/3 & -4/3 \\
0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
3 & 4 & 8 & 24 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

12. \[
\begin{bmatrix}
1/3 & -4/3 & -8/3 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
3 & 4 & 8 & 24 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

13. \[
\begin{bmatrix}
1 & 0 & 0 \\
-2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

14. \[
\begin{bmatrix}
1 & 0 & -1 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & -1 \\
-2 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\]

15. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & -1 \\
-2 & 0 & 1 \\
0 & 0 & 1
\end{bmatrix}
\]

16. \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & -1 \\
0 & 0 & 1
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 0 & -1 \\
0 & 0 & 1 \\
-2 & 1 & 0
\end{bmatrix}
\]
17. \[
\begin{bmatrix}
1 & 0 & -1 \\
2 & -1 & 0 \\
-2 & 1 & 0
\end{bmatrix}
\cdot
\begin{bmatrix}
1 & 1 & 1 & 1 \\
2 & 0 & 1 & 4 \\
0 & 1 & 1 & 2
\end{bmatrix}
\]

18. Solve the system of equations
\[
\begin{align*}
x + y + z &= 1 \\
2x + z &= 4 \\
y + z &= 2
\end{align*}
\]
Can you relate your solution to the previous several computations?