General Information. The primary purpose of this assignment is to help you learn and internalize a lot of material that will be helpful in your other courses, and subsequent professional life. The questions here contain constructions and examples that everyone should know.

Some of you will seek help from those who already know the material. This is acceptable, but indicate the name(s) of the people with whom you spoke. If you get the answers from someone else and if you don’t bother to understand them, then you are only hurting yourself. Grades don’t matter. Understanding does!

Having written these admonitions, let me turn to the instructions. Each problem discusses a particular example of a relation. The problem will guide you through the proofs that the relation is a total order, partial order, or an equivalence relation. Other properties of the relations will be discussed. Work through as many of the details as you can. If you can’t verify a property, assume that it is true to verify other properties.

Good luck.

Definitions

Let X, Y, and Z denote sets.

A. A relation from X to Y denoted \( Y \overset{R}{\rightarrow} X \) is a subset of \( X \times Y = \{(x, y) : x \in X \land y \in Y\} \). If \((a, b) \in R\), we usually write \( aRb \).

B. The composition of relations \( Z \overset{S}{\rightarrow} Y \) and \( Y \overset{R}{\rightarrow} X \) is the set \( S \circ R = \{(x, z) : \exists y \in Y \text{ with } ySz \land xRy\} \).

C. The identity relation on a set X is the relation \( X \overset{I}{\rightarrow} X \) given by \( I = \{(x, x) : x \in X\} \).

D. A relation \( X \overset{R}{\rightarrow} X \) is said to be reflexive if and only if \( \forall x \in X \text{ we have } xRx \).

E. Let \( x, y \in X \). A relation \( X \overset{R}{\rightarrow} X \) is said to be symmetric if and only if \( xRy \implies yRx \).

F. Let \( x, y \in X \). A relation \( X \overset{R}{\rightarrow} X \) is said to be anti-symmetric if and only if whenever \( xRy \land yRx \), then \( x = y \).

G. Let \( x, y, z \in X \). A relation \( X \overset{R}{\rightarrow} X \) is said to be transitive if and only if whenever \( xRy \land yRz \), then \( xRz \).

H. A relation \( X \overset{R}{\rightarrow} X \) is said to be total if and only if \( \forall x, y \in X \text{ either } xRy \text{ or } yRx \).

I. The inverse of a relation \( Y \overset{R}{\rightarrow} X \) is the relation \( X \overset{R^{-1}}{\leftarrow} Y \) given by \( R^{-1} = \{(y, x) : xRy\} \).
Questions

1. Show that the identity relation \( X \xleftarrow{I} X \) is an identity under composition. That is show
   (a) if \( Y \xleftarrow{R} X \) is any relation, then \( R \circ I = R \), and
   (b) if \( X \xleftarrow{S} Y \) then \( I \circ S = S \).

2. Show that composition of relations is associative. That is assume that \( X, Y, Z, \) and \( W \) are sets, and \( W \xleftarrow{T} Z, Z \xleftarrow{S} Y, \) and \( Y \xleftarrow{R} X \) are relations. Then show
   \((T \circ S) \circ R = T \circ (S \circ R)\).

3. Let \( Y \xleftarrow{R} X \) denote a relation. Show that \( R^{-1} \circ R = I_X \) and that \( R \circ R^{-1} = I_Y \), where \( X' \) and \( Y' \) denote the domain and range. Note: the identity relation is defined on a set. Thus \( I_Y = \{(y, y) : y \in Y\} \) and \( I_X = \{(x, x) : x \in X\} \) represent different subsets.

4. Let \( X = \{1, 2\} \). Find a relation \( X \xleftarrow{R} X \) that is symmetric but not reflexive.

5. Let \( X = \{1, 2\} \). Find a relation \( X \xleftarrow{R} X \) that is reflexive but not symmetric.

6. Define a relation \( N \xleftarrow{\mid} N \) by \( a \mid b \) (pronounced “\( a \) divides \( b \)”) if and only if \( \exists k \in N \) such that \( b = kn \). Show that \( \mid \) defines a partial order on \( N \). That is show that \( \mid \) is reflexive, anti-symmetric and transitive (D, F, and G above).

7. The relation \( a \) divides \( b \) (written above as \( a \mid b \)) can also be defined over the integers, \( \mathbb{Z} \): \( a \mid b \) if and only if \( \exists k \in \mathbb{Z} \) such that \( b = ka \). Is this a partial order on \( \mathbb{Z} \)? Why or why not?

8. In the next set of problems, we will construct the integers modulo \( n \). Suppose that \( n \) is a non-negative integer. Thus \( n \in \mathbb{Z}^+ \). Define a relation \( \mathbb{Z} \xrightarrow{\equiv_n} \mathbb{Z} \) by the formula \( a \equiv_n b \) if and only if \( n|\overline{b-a} \).
   (a) Show that \( \equiv_n \) is a reflexive relation.
   (b) Show that \( \equiv_n \) is a symmetric relation.
   (c) Show that \( \equiv_n \) is a transitive relation.
   (d) A partition of a set \( X \) is a family \( \mathcal{F} \) of subsets of \( X \) such that
      i. \( X = \bigcup_{A \in \mathcal{F}} A \),
      ii. if \( A, B \in \mathcal{F} \) and \( A \neq B \), then \( A \cap B = \emptyset \).
      iii. if \( A \in \mathcal{F} \), then \( A \neq \emptyset \).
   Let \( j \in \mathbb{Z} \). Let \( A_j = \{a \in \mathbb{Z} : a \equiv_n j\} \). Show that \( \mathcal{F} = \{A_j : j \in \{0, 1, 2, \ldots, n-1\}\} \) is a partition of \( \mathbb{Z} \). The sets \( A_j \) are called equivalence classes.
(e) Show that, as sets, $A_i = A_j$ if and only if $i \equiv n \cdot j$. The set $A_j$ will be denoted by $[j]$ and is called an equivalence class. The element $j \in \mathbb{Z}$ is called a representative of the equivalence class.

(f) Suppose that $n = 5$. List the elements of the equivalence classes $[0],[1],[2],[3],[4]$.

(g) Again for $n = 5$, in which class (among $[0],[1],[2],[3],[4]$) is the integer 6, 557, 896, 789, 254, 353, 534, 538 found?

(h) Let $n$ return to denoting an arbitrary integer. Define $[i] \oplus [j] = [i + j]$. You must show that the operation $\oplus$ is well-defined. That is, you must show that if $[i] = [i']$ and $[j] = [j']$, then $[i + j] = [i' + j']$.

(i) Similarly, define $[i] \otimes [j] = [i \times j]$. Show that the operation $\otimes$ is well-defined in the sense that if $[i] = [i']$ and $[j] = [j']$, then $[i \times j] = [i' \times j']$.

Please note, the symbols $\oplus$ and $\otimes$ are only temporary symbols used for these problems and have different standard usages within the rest of the mathematical literature.

(j) Show that $[0]$ serves as the additive identity for $\oplus$. That is, show that $\forall j \in \mathbb{Z}$, we have $[0] + [j] = [j]$.

(k) Show that $[1]$ serves as the multiplicative identity for $\otimes$. That is, show that $\forall j \in \mathbb{Z}$, we have $[1] \otimes [j] = [j]$.

(l) Construct an addition table and a multiplication table for $n = 5$.

(m) Construct an addition table and a multiplication table for $n = 6$.

(n) Looking at the multiplication tables for $n = 5$ and $n = 6$, what significant difference do you see among the results.

9. This problem concerns polynomials, divisibility, and a similar equivalence relation to that above. A polynomial of degree $n$ in a variable $x$ with real coefficients is the set of expressions of the form

$$f(x) = \sum_{j=0}^{n} a_j x^j = a_0 + a_1 x + \cdots + a_n x^n$$

where $n \in \{0, 1, 2, \ldots \} = \mathbb{N}$, $a_j \in \mathbb{R}$ for $j \in \{0, 1, 2, \ldots, n\}$, and $a_n \neq 0$. The set of all polynomials is the set

$$\mathbb{R}[x] = \{ \sum_{j=0}^{n} a_j x^j = a_0 + a_1 x + \cdots + a_n x^n : n \in \mathbb{N}, a_j \in \mathbb{R} \text{ for } j \in \{0, 1, 2, \ldots, n\} \& a_n \neq 0 \}.$$ 

(a) Show that degree defines a partial order on the set of polynomials. That is say that $f(x) \prec g(x)$ if and only if $\deg(f(x)) \leq \deg(g(x))$. Now show that the relation $\prec$ is a reflexive, antisymmetric, transitive relation on $\mathbb{R}[x]$. 

3
(b) What is the set of polynomials of degree 0?

(c) The notion of degree allows a division algorithm to be defined on \( \mathbb{R}[x] \). Namely, given polynomials \( f(x) \) and \( g(x) \), with \( \deg(f(x)) < \deg(g(x)) \), there are unique polynomials \( q(x) \) and \( r(x) \) such that \( g(x) = q(x)f(x) + r(x) \) where \( \deg(r(x)) \leq \deg(f(x)) \), and \( \deg(q(x)) \cdot \deg(f(x)) = \deg(g(x)) \). The polynomial \( f(x) \) is called the divisor, \( g(x) \) is called the dividend, \( q(x) \) is called the quotient, and \( r(x) \) is called the remainder.

(d) For polynomials \( f(x), g(x) \in \mathbb{R}[x] \), say that \( f(x) | g(x) \) (that is \( f(x) \) divides \( g(x) \)) if there is a polynomials \( q(x) \) such that \( g(x) = f(x)q(x) \). Thus the remainder of \( g(x) \) upon division by \( f(x) \) is 0. Show that \( | \) is a reflexive, transitive relation on \( \mathbb{R}[x] \).

(e) For polynomials \( f(x), g(x) \in \mathbb{R}[x] \), define \( f(x) \equiv g(x) \) provided \( (x^2 + 1) \) divides \( g(x) - f(x) \). Show that \( \equiv \) is an equivalence (reflexive, symmetric, transitive) relation on \( \mathbb{R}[x] \).

(f) Show that any \( f(x) \in \mathbb{R}[x] \) is equivalent (using the relation \( \equiv \)) to a polynomial of the form \( a + bx \) where \( a, b \in \mathbb{R} \).

(g) Denote by
\[
\langle a, b \rangle = \{ f(x) \in \mathbb{R}[x] : f(x) \equiv a + bx \}.
\]

(h) Show that the operation
\[
\langle a, b \rangle \cdot \langle c, d \rangle = \langle ac - bd, bc + ad \rangle
\]
agrees with the multiplication \([f(x)] \cdot [g(x)] = [f(x)g(x)]\).

(i) This last problem is a bonus problem for all who are following the thread. Show that the equivalence classes can be added and multiplied in the obvious way, and that with these operations the set of equivalence classes is the same as the complex numbers.