Coloring Knot Diagrams, Knotted Surfaces, and Quandles

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Fox Colorings

(a,b) \begin{pmatrix} 0 & t \\ 1 & 1-t \end{pmatrix}

Braid Example
Burau Matrices and Alex. Polyn.

\[
\begin{pmatrix}
\frac{(1-t)^2}{t^2} & t & 1 + \frac{(1-t)^2}{t^3} - t - \left( \frac{1-t}{t^3} \right) + \frac{1-t}{t} \\
-t\left( \frac{1-t}{t} \right) & 0 & -\left( \frac{1-t}{t^2} \right) \\
t & (1-t) t^2 & (1-t)^2 t \\
0 & (1-t) t^2 & t + (1-t)^2 t \\
\end{pmatrix}
\]

\[
\begin{pmatrix}
4 & -1 & -2 & 0 \\
2 & 0 & -2 & 1 \\
-1 & 2 & -4 & 4 \\
0 & 2 & -5 & 4 \\
\end{pmatrix}
\begin{pmatrix}
b \\
a \\
a \\
b \\
\end{pmatrix}
= 
\begin{pmatrix}
-3a + 4b \\
-2a + 3b \\
-2a + 3b \\
-3a + 4b \\
\end{pmatrix}
\]
• $\sigma$ — an $n$-string braid.

• $B(\sigma)$ — the $n \times n$ matrix exemplified above.

• Consider the matrix of $(n-1) \times (n-1)$ minors of $B(\sigma) - I$

• $\Delta(\sigma)$ any one entry of this matrix.

• For Example, $\Delta(t) = 2 - 3t + 3t^2 - 3t^3 + 2t^4$

• NB: $\Delta$ is well def’ed. up to $\pm t^{\pm 1}$.

• $\Delta(t)$ is called the Alexander Polynomial of the knot $\hat{\sigma}$. 
\[\Delta(-1)\] called the determinant of the knot, related to possible colorings.
Quandles

A QUANDLE is a set, $Q$, with a binary operation $\ast$ defined such that

- $a \ast a = a$

- $\forall a, b \in Q \ \exists! c \in Q \ \text{s.t.} \ a = c \ast b$

- $(a \ast b) \ast c = (a \ast c) \ast (b \ast c)$
Examples:
• \( a \ast b = ta + (1 - t)b \)

• at \( t = -1 \), \( a \ast b = 2b - a \) — read modulo \( n \)

• More generally, \( S \subset G \) — a group. \( bab^{-1} \in S \) if \( a, b \in S \). On \( S \) define \( a \ast b = bab^{-1} \).
a*b = rotate a counter clockwise about the vertex b

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The example points to the possibility of Quandle Extensions.

- $S$ is a set; $X$ is a quandle

- $\alpha : X \times X \to S^{S \times S}$
• Define a product $*$ on $S \times X$ via

$$(s, x) * (t, y) = (\alpha_{(x,y)}(s, t), x * y).$$

• Direct calculation shows

$$((s, x) * (t, y)) * (u, z)$$

$$= ((s, x) * (u, z)) * ((t, y) * (u, z))$$

$$\Leftrightarrow$$

$$\alpha_{(x*y, z)}(\alpha_{(x, y)}(s, t), u)$$

$$= \alpha_{(x*z), (y*z)}(\alpha_{(x, z)}(s, u), \alpha_{(y, z)}(t, u))$$

• This can be verified via Reid. moves.

• $\Rightarrow$ there is a cohomology theory of quandles analogous to group cohomology.
Counting Colors

• Important knot invariant: the number of colorings by a fixed quandle.

• ¿Why important? Graña and Pregel, Lopes have used colorings and generalizations from cohom. thy to distinguish many classical knots and knotted surfaces.

• One way to count

— $V$ a vector space with basis $X$, a quandle.

— $R_{c,d}^{a,b}$ a map $V^\otimes 2 \to V^\otimes 2$

— $R_{c,d}^{a,b} = \begin{cases} 1 & \text{if } b = c \ \& \ d = a \ast b \\ 0 & \text{else} \end{cases}$
Then quandle rule III ⇒ $R$ satisfies the YBE:

$$R_{d,e}^{a,b} R_{g,h}^{e,c} R_{i,j}^{d,g} = R_{d,e}^{b,c} R_{i,g}^{a,d} R_{j,h}^{g,e}$$

Then the number of colorings is evaluated as a state-sum invariant using this $R$ matrix.

$R_{c,d}^{a,b}$ can be written as $\phi(a, b)$. Then

$$\phi(a, b)\phi(a*b, c)\phi(b, c) = \phi(b, c)\phi(a, c)\phi(a*c, b*c)$$

**Observations**

— This is a cocycle condition similar to that for $\alpha$ def’d above.

— There is a cohom. thy [FRS] for which knot invariants can be constructed.

— An appropriate version def’s inv. for knotted surfaces.