Basic Definitions

Be able to state precisely the definitions of the terms specified below. The terms to be defined appear in boldface. The definitions that I expect you to reproduce appear beside the terms.

1. **Subspace of a Vector Space.** A subset \( W \subset V \) of a vector space, \( V \), is said to be a **subspace** if
   - \( 0 \in W \)
   - \( \alpha v + \beta w \in W \) whenever \( \alpha, \beta \in \mathbb{R} \) and \( v, w \in W \).

2. **The Span of a set of vectors.** Let \( S = \{ \vec{v}_1, \ldots, \vec{v}_k \} \subset V \) where \( V \) is a vector space. The **span** of \( S \), \( \text{Span}(S) = \{ \sum_{j=1}^{k} \alpha_j \vec{v}_j : \alpha_j \in \mathbb{R} \text{ for } j = 1, \ldots, k \} \), is the set of all linear combinations of the elements of \( S \).

3. **A linearly independent set of vectors.** The set \( S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is said to be **linearly independent** if whenever
   \[
   \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2 + \cdots + \alpha_k \vec{v}_k = 0
   \]
   it follows that
   \[
   \alpha_1 = \alpha_2 = \cdots = \alpha_k = 0.
   \]

4. **A basis for a vector space.** The set \( S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) is said to be a **basis** for a vector space \( V \), if
   - \( S \) spans \( V \), and
   - \( S \) is a linearly independent set.

Computational questions

1. Solve the system of equations:
   \[
   \begin{align*}
   x + y + z &= 1 \\
   y &= 0
   \end{align*}
   \]

2. Solve the system of equations
   \[
   \begin{align*}
   x + y + z &= 1 \\
   2x + y + z &= 3 \\
   5x + 3y + 3z &= 6
   \end{align*}
   \]
3. Write the equation of the plane

\[3x + 2y - 4z = 12\]

in parametric form.

4. Let

\[ A = \begin{bmatrix} 2 & -5 \\ 3 & 1 \end{bmatrix} \]

Compute \( A^2 \) and \( A^3 \).

5. The reduced row echelon form of the matrix \( A \) that is associated to the homogeneous system of equations

\[
\begin{align*}
5x &+ 4y - 2z = 0 \\
x &+ 2y + 2w = 0
\end{align*}
\]

is

\[
\begin{bmatrix} 1 & 0 & -\frac{2}{3} & -\frac{4}{3} \\ 0 & 1 & \frac{1}{3} & \frac{5}{3} \end{bmatrix}.
\]

Determine the solution set.

6. Give the set of solutions to the system of equations:

\[
\begin{align*}
x &+ y + z = 0 \\
2x &+ y - 3z = 0
\end{align*}
\]

7. Determine if the following system of equations is consistent (has a solution). If so write the set of solutions in standard form:

\[
\begin{align*}
x + y + z + w &= 1 \\
2x + y - z - 3w &= 2 \\
x + y + 3z &= 1 \\
x - 2y + z + 4w &= 4
\end{align*}
\]

8. Solve the system of equations.

\[
\begin{align*}
x &+ 2y + 4z - 5t = 3 \\
3x &- y + 5z + 2t = 4 \\
5x &- 4y + 6z + 9t = 2
\end{align*}
\]

9. Solve the system of equations:

\[
\begin{align*}
x &- y + 2z = 0 \\
2x &+ y + z = 0 \\
5x &+ y + 4z = 0
\end{align*}
\]
10. Write the matrix

\[
A = \begin{bmatrix} 1 & 3 \\ -2 & 4 \end{bmatrix}
\]

as a product of elementary matrices. Recall that a matrix is an elementary matrix if it is obtained from the identity matrix by means of an elementary row operation.

11. Find the equation of the plane that is parallel to \(4x + 3y - 2z = 11\) and that contains the point \(Q(2, -1, 3)\).

12. Determine if the set of vectors indicated is linearly independent.

(a) \[
\left\{ \begin{bmatrix} -1 \\ 3 \\ 0 \\ 7 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 5 \\ 12 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 1 \\ -8 \end{bmatrix}, \begin{bmatrix} 4 \\ -2 \\ 1 \\ 6 \end{bmatrix} \right\}
\]

(b) \[
\left\{ \begin{bmatrix} -1 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix} \right\}
\]

13. Express the matrix

\[
M = \begin{bmatrix} 4 & 7 \\ 7 & 9 \end{bmatrix}
\]

as a linear combination of the matrices \(A, B,\) and \(C\) where

\[
A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 4 & 5 \end{bmatrix}.
\]