1. Definitions and Theorems.

(a) Be able to define the definite integral:

\[ \int_a^b f(x) \, dx = \lim_{N \to \infty} \sum_{j=1}^{N} f(x_j^*)(\Delta x) \]

where \( \Delta x = (b - a)/N \) and \( x_j^* \in [x_{j-1}, x_j] \).

(b) Be able to state the Fundamental Theorem of Calculus:

I. Let \( y = f(t) \) denote a continuous function that is defined on a closed interval \([a, b]\).

Define

\[ A(x) = \int_a^x f(t) \, dt. \]

Then

\[ A'(x) = f(x). \]

Or

II. Let \( y = f(t) \) denote a continuous function that is defined on a closed interval \([a, b]\).

Suppose that \( G'(x) = f(x) \). Then

\[ \int_a^b f(x) \, dx = G(b) - G(a). \]

(c) Be able to define the limit of a sequence:

\[ \lim_{n \to \infty} a_n = L \]

if and only if for every \( \epsilon > 0 \) there is an integer \( N \) such that if \( N \leq n \) we have

\[ |a_n - L| < \epsilon. \]

2. Integrals

(a) Compute the area that lies between the curves:

i. \( y = x^3 - x^2 + 1 \) and \( y = 1. \)
ii. \( y = x^2 \) and \( y = \sqrt{x} \) for \( x \in [0, 2]. \)
iii. \( y = \sin(x) \) and \( y = \cos(x) \) for \( x \in [0, \frac{\pi}{2}] \)

(b) i. \( \int_0^1 (x^3 + x^{1/3}) \, dx \)
ii. \( \int_1^2 \frac{1}{x^2} \, dx \)
iii. \( \int_1^4 (3x^4 + 5x^3 - 4x^2 + 7) \, dx \)
iv. \( \int \frac{1}{x^3} \, dx \)

(c) i. \( \int_0^\pi (1 + \cos(x)) \, dx \)
ii. \( \int e^x \cos(e^x) \, dx \)
iii. $\int x^2 e^{x^3} \, dx$
iv. $\int \sin (x) \cos^5 (x) \, dx$

(d) i. $\int xe^x \, dx$
ii. $\int x^2 \sin (2x) \, dx$
iii. $\int e^x \cos x \, dx$
iv. $\int x \ln (x) \, dx$
v. $\int \arcsin (x) \, dx$
vi. $\int \arctan (x) \, dx$

(e) i. $\int \sec^4 (x) \, dx$
ii. $\int \sin^5 (x) \, dx$
iii. $\int \sin^2 (x) \cos^2 (x) \, dx$
iv. $\int 2 \sin (x) \cos (x) \, dx$

(f) i. $\int \frac{dx}{\sqrt{36-x^2}}$
ii. $\int \frac{dx}{\sqrt{9+x^2}}$
iii. $\int \frac{dx}{x\sqrt{x^2-25}}$
iv. $\int \frac{x \, dx}{x^2+49}$

(g) i. $\int \frac{1}{(x+1)(x-1)^2} \, dx$
ii. $\int \frac{1}{(x+1)(x-1)} \, dx$
iii. $\int \frac{1}{(x^2+1)(x-1)} \, dx$
iv. $\int \frac{dx}{x^2+5x-6}$
v. $\int \frac{(3x-5) \, dx}{(x-2)(x+3)}$

(h) i. $\int_0^1 \frac{dx}{\sqrt{x}}$
ii. $\int_0^\infty 2xe^{-x^2} \, dx$
iii. $\int_1^\infty \frac{1}{(x^{3/4})} \, dx$
iv. $\int_0^\infty \frac{1}{(x^{1/4}+1)} \, dx$
v. $\int_0^\infty \frac{1}{2} e^{-x/2} \, dx$

3. (a) Compute the volume obtained by rotating the region bounded by the curves $y = x^2$, $y = 0$, $x = 1$ about the x-axis.

(b) Compute the volume obtained by rotating the region bounded by the curves $y = 2x$, $y = 0$, and $x = 2$ about the y-axis.

(c) Compute the volume obtained by rotating the region bounded by the curves $y = e^{-x}$, $y = 0$, $x = 0$, $x = 1$ about the x-axis.

d) Compute the volume obtained by rotating the region bounded by the curves $y = \sqrt{2}$, $x = y^2$, and $x = 0$ about the x-axis.
(e) Find the volume of the tetrahedron, by considering the triangular cross sections perpendicular to the $x$-axis.

4. (a) Find the length of the curve $y = x^{\frac{3}{2}}$ from $x = 0$ to $x = 4$. Recall that the formula for arc-length is $L = \int_{a}^{b} \sqrt{1 + (f'(x))^2} \, dx$.

(b) Compute the length of the curve of the from $0 \leq x \leq 3$ of the function

$$y = \frac{1}{3}(x^2 + 2)^{\frac{3}{2}}$$

(c) Find the area of the surface generated by revolving the curve $x = \frac{e^{y} + e^{-y}}{2}$, for $0 \leq y \leq \ln(2)$, about the $y$-axis.

(d) Find the surface area that is obtained by rotating $y = \sqrt{64 - x^2}$ about the $x$-axis for $x \in [0, 8]$.

5. (a) A force of 2-Newton will stretch a rubber band $\frac{2}{100}$ meters. Assuming that Hooke’s Law applies, how far with a force of 4-Newton stretch the rubber band? How much work does it take to stretch the rubber band this far? Recall: Hooke’s law states that the force required to stretch a spring is proportional to the length that it is stretched.

(b) A force of 25-Newton will stretch a spring 5 meters beyond its natural length. Assuming that Hooke’s Law applies, how much work does it take to stretch the spring 10 meters beyond its natural length? Recall: Hooke’s law states that the force required to stretch a spring is proportional to the length that it is stretched.
(c) The half-life of the plutonium isotope is 24,360 years. If 10 grams of plutonium is released into the atmosphere by a nuclear accident, how many years will it take for 80% of the isotope to decay?

(d) A leaky bucket is being lifted 20 meters. It originally holds 10 kilograms of water, and at the end of its journey it holds 5 kilos. How much work does it take to lift the bucket?

6. Determine which of the following sequences converge. If the sequence does converge, find the limit.

(a) \( a_n = 1 + \left( \frac{1}{3} \right)^n \)
(b) \( a_n = \frac{n^2 + n + 3}{3n^2 + 2n + 1} \)
(c) \( a_n = \frac{n^3}{n^3 + 3n^2 - 2} \)
(d) \( a_n = \frac{n \cos(\pi n)}{n} \)
(e) \( a_n = \ln(n) - \ln(n + 1) \)
(f) \( a_n = (-1)^n + \frac{1}{n} \)
(g) \( a_n = (1 - \frac{2}{n})^n \)
(h) \( a_n = \left( \frac{n}{n + 3} \right)^n \)
(i) \( a_n = \left( \frac{-8}{n} \right)^n \)
(j) \( a_n = \sqrt[5]{5n} \)

7. For each of the series determine a formula for the \( n \)th partial sum:

(a) \( \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{(n + 1) \cdot (n + 2)} + \cdots \)

(b) \( 1 + \frac{1}{5} + \frac{1}{25} + \cdots + \frac{1}{5^n} + \cdots \)

(c) \( \frac{9}{10} + \frac{9}{100} + \cdots + \frac{9}{10^n} + \cdots \)

(d) \( \left( -\frac{2}{3} \right)^2 + \left( -\frac{2}{3} \right)^3 + \left( -\frac{2}{3} \right)^4 + \cdots \)

8. Use any test that you like to determine if the given series converges.

(a) \( \sum_{n=2}^{\infty} \frac{\ln n}{\sqrt{n}} \)

(b) \( \sum_{n=1}^{\infty} \frac{1}{n^{100,000} / 100,000} \)
(c) \[ \sum_{n=1}^{\infty} \frac{n}{n^2 + 1} \]

(d) \[ \sum_{n=1}^{\infty} \frac{1}{3^n - 1} \]

(e) \[ \sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2 + 3} \]

(f) \[ \sum_{n=1}^{\infty} \frac{1}{n!} \]

(g) \[ \sum_{n=1}^{\infty} \left(1 + \frac{3}{n}\right)^n \]

(h) \[ \sum_{n=1}^{\infty} \frac{n!}{(2n)!} \]

(i) \[ \sum_{n=1}^{\infty} \frac{1}{n^{2/3}} \]

(j) \[ \sum_{n=2}^{\infty} \frac{1}{n \sqrt{n^2 - 1}} \]

(k) \[ \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n \]

(l) \[ \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}} \]

(m) \[ \sum_{n=1}^{\infty} \frac{n^2(n + 2)!}{n!3^{2n}} \]

(n) \[ \sum_{n=1}^{\infty} \frac{1}{n^{1.1}} \]
\[ \sum_{n=1}^{\infty} \left( 1 + \frac{-2}{n} \right)^n \]

\[ \sum_{n=0}^{\infty} \frac{n}{n^2 + 1} \]

\[ \sum_{n=0}^{\infty} \frac{(-3)^n}{n!} \]

9. Compute the interval of convergence for the series:

(a) \[ \sum_{n=0}^{\infty} \frac{(x - 2)^n}{3^n} \]

(b) \[ \sum_{n=0}^{\infty} \frac{(-1)^n(x + 6)^n}{n \cdot 3^n} \]

(c) \[ \sum_{n=1}^{\infty} nx^n \]

(d) \[ \sum_{n=1}^{\infty} \frac{x^n}{(2n)!} \]

10. Find the 4th Taylor polynomial, \( P_4 \), generated by \( f(x) = \frac{1}{x} \) at center \( a = 2 \)

11. Use substitution and power series operations to find the Taylor series about \( x=0 \) of the following functions:

(a) \( e^{-\frac{x}{2}} \)

(b) \( \sin \left( \frac{\pi x}{2} \right) \)

(c) \( x^2 \cos (3x) \)

(d) \( \frac{1}{3-x} \)

(e) \( \ln(1 + \frac{3}{4}x) \)

12. Compute the parametrization of the line segment from \((1, 2)\) to \((3, -8)\) for \( t \in [0, 1] \).

13. Give a parametrization of the ellipse \( \frac{x^2}{25} + \frac{y^2}{9} = 1 \) that travels once counter-clockwise in an interval \( t \in [0, 2\pi] \).

14. Give the equation of the tangent line to the curve at the given point
(a) \[ x(t) = 2 \cos(t) \quad y(t) = 2 \sin(t) \quad \text{at } t = \pi/6. \]

(b) \[ x(t) = 2t^2 + 3 \quad y(t) = t^4 \quad \text{at } t = -1. \]

(c) \[ x(t) = t + e^t \quad y(t) = 1 - e^t \quad \text{at } t = 0. \]

15. areas of polar graphs (and the graphs of the functions) such as

(a) \( r = 1 + \cos(\theta) \)

(b) one petal of \( r = \cos(3\theta) \)

(c) \( r = \sin(\theta) \) for \( \theta \in [0, \pi] \).