Some Properties of the Projections of Knotted Surfaces

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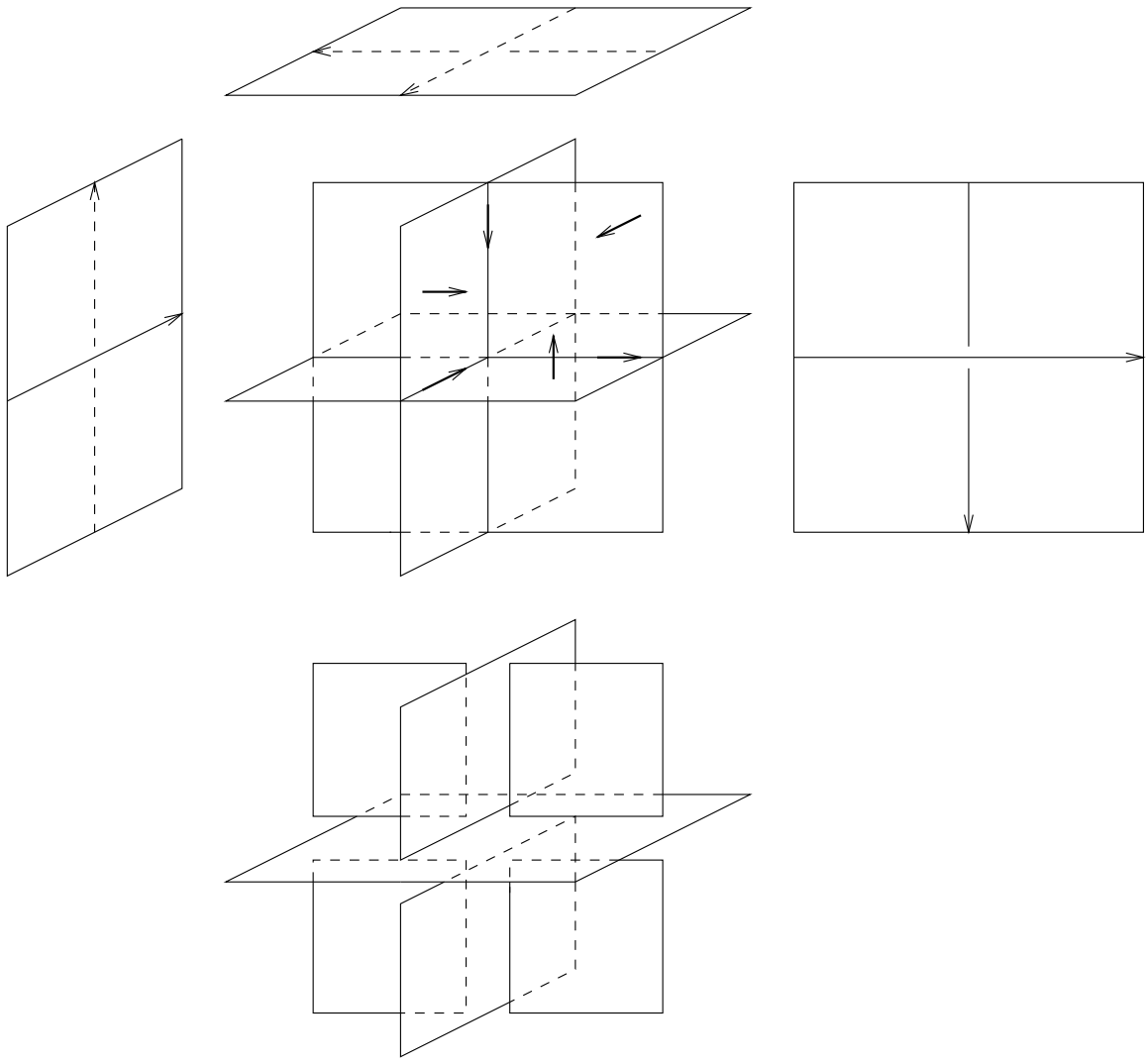
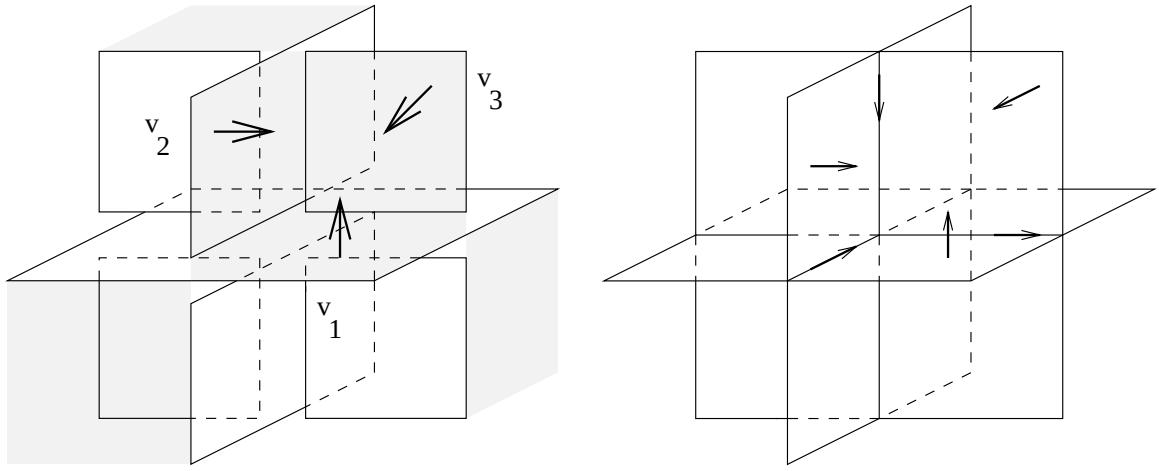


Orientations of triple points



Orientations an Colorings

Definitions: Let $p : F \to \mathbf{R}^3$ denote the projection of a surface in 4-space.

The *colored sign of a triple point* (of an unoriented surface) of p is defined to be the oriented triple (V_T, V_M, V_B) of normal vectors to the top, middle, and bottom sheets where these vectors are chosen to point into a black region near a triple point.

The *sign* of a triple point of an oriented surface is defined to be the oriented triple (V_T, V_M, V_B) of vectors normal to the top, middle, and bot-

tom sheet where these vectors are chosen to give a co-orientation for $p(F)$ so that the tangent orientation plus the co-orientation agrees with that of 3-space.

The *color of a triple point* of an *oriented surface* is the color of the prefered region into which the vectors V_T, V_M, and V_B point. The *color of a double point arc* is the color of the region into which the oriented normals of intersecting sheets point. *Distinguished edges* at a triple point are the 3 edges (among 6) that 'bound' the region into which the normals point.

Theorem [CS, Proc AMS 1997]: Let $T = T(+) - T(-)$ where $T(\pm)$ denotes the number of $\pm$ colored triple points. Then for any surface in 4-space that projects to an immersion

$$T = 0.$$

Lemma [CCS] A triple point has a positive colored orientation if and only if it is either (black, positive) or (white, negative).

Notation T(color, sign) denotes the number of triple points with given color and oriented sign.

Proposition The following hold at every triple point of an oriented surface.

(1) All distinguished edges have the same color, this color coincides with the color of the triple point; the remaining 3 edges have the opposite color.

(2) A triple point is positive if and only if among the three distinguished edges, the only edge that points towards the triple point is the edge that is formed by the top and bottom sheets.

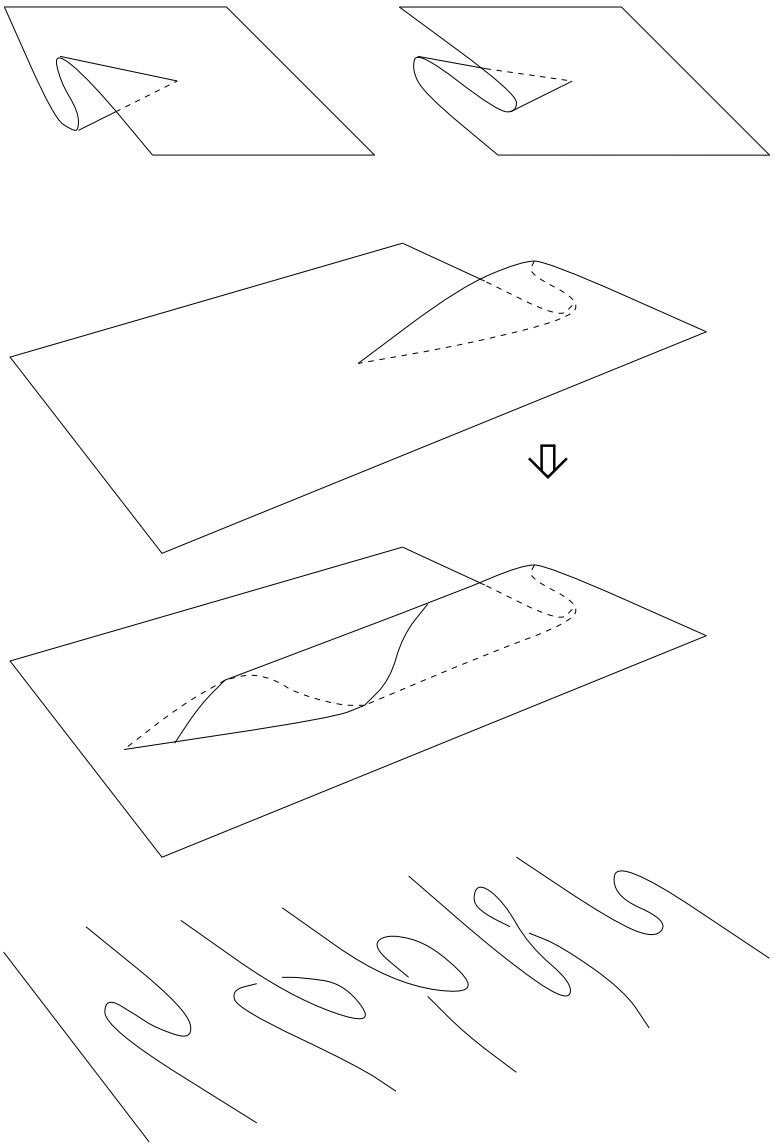
Theorem [CCS] For an immersed diagram of an oriented knotted surface,

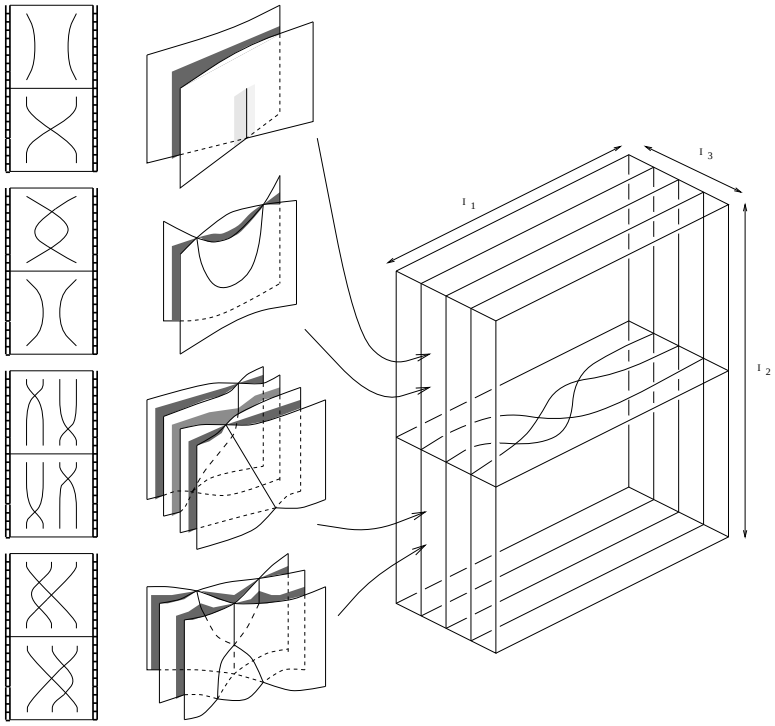
$$T(B,+) + T(W,+) = T(B,-) + T(W,-).$$

In other words, the number of positive (uncolored) triple points equals the number of (uncolored) negative triple points.

Theorem [CCS] (a) Let $K : F \to \mathbf{R}^4$ be a knotted surface. Then K is equivalent to a knotted surface which projects to one with either one or no cusps depending on whether the Euler characteristic of F is odd or even, respectively.

(b) Let $K : F \to \mathbf{R}^4$ denote a knotted oriented surface. The K is a equivalent to a surface that projects to one with neither branch points nor cusps.

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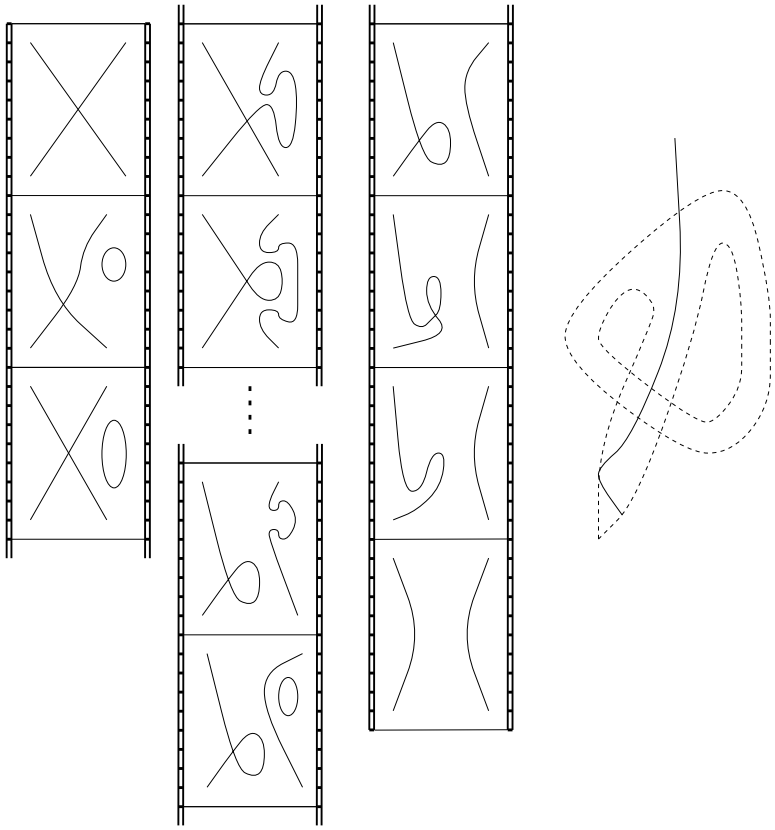

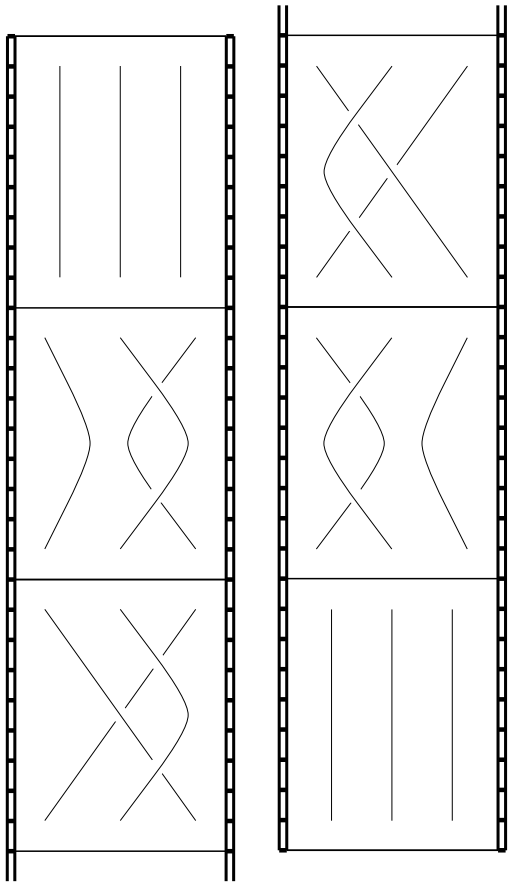

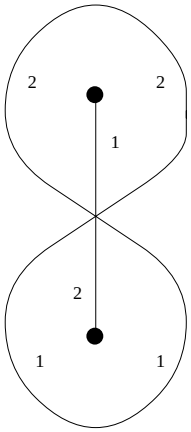
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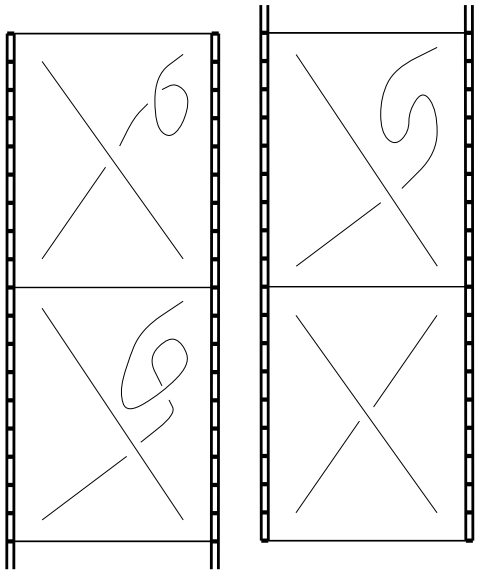

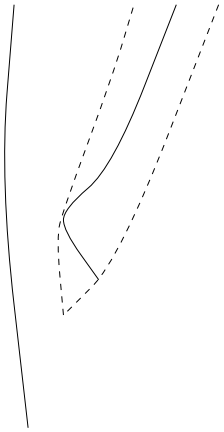

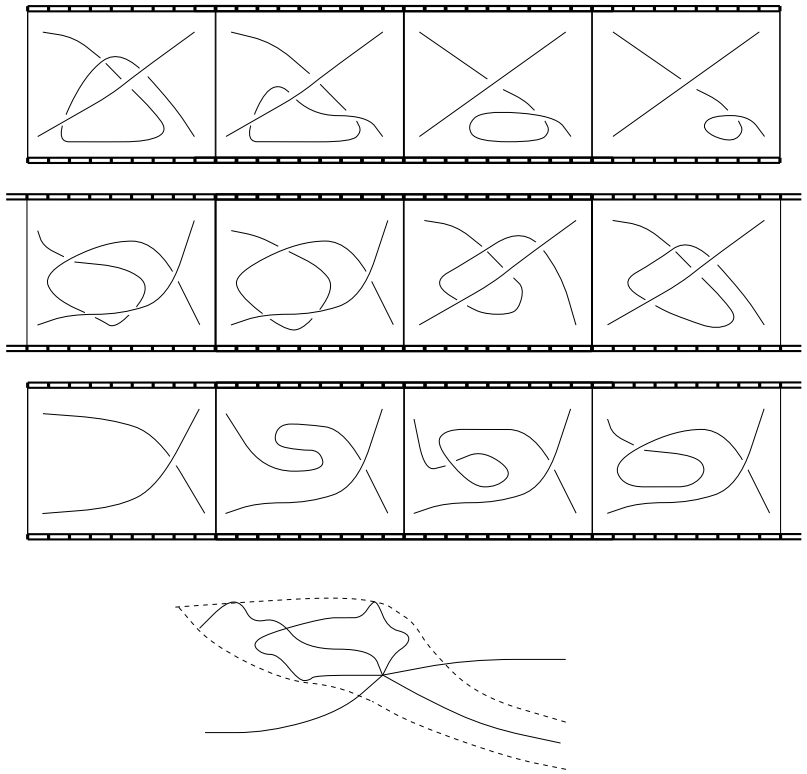

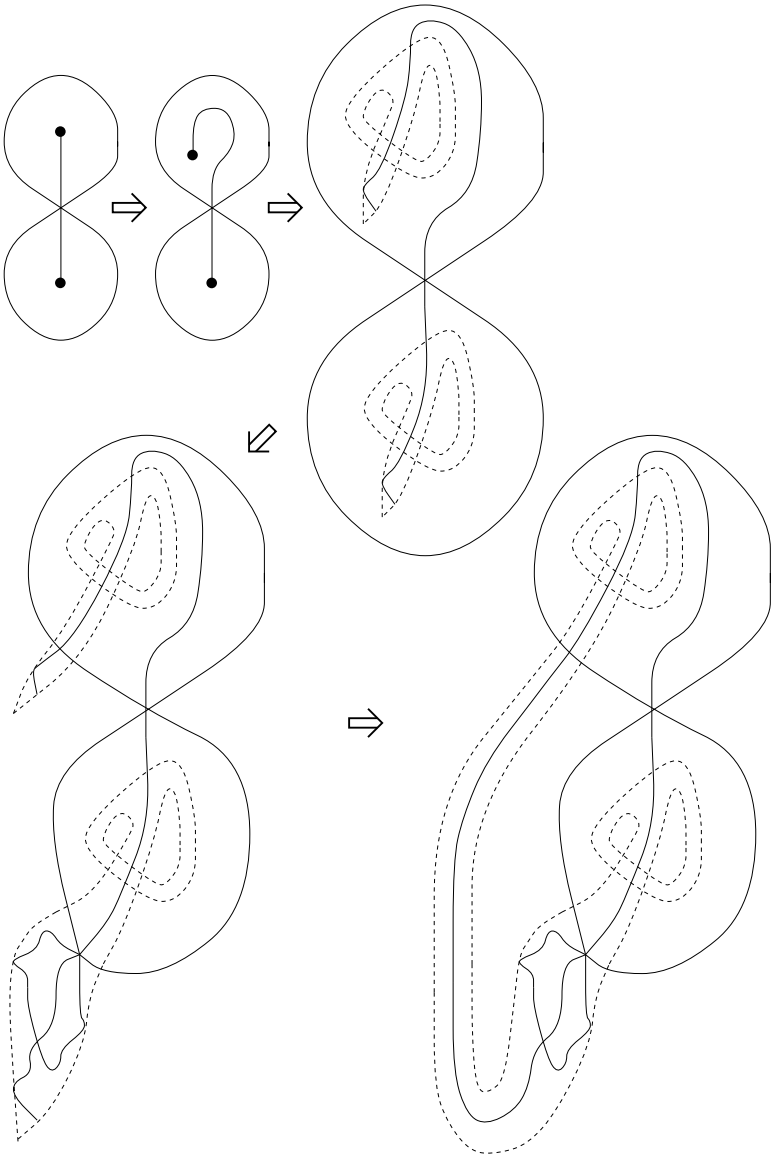

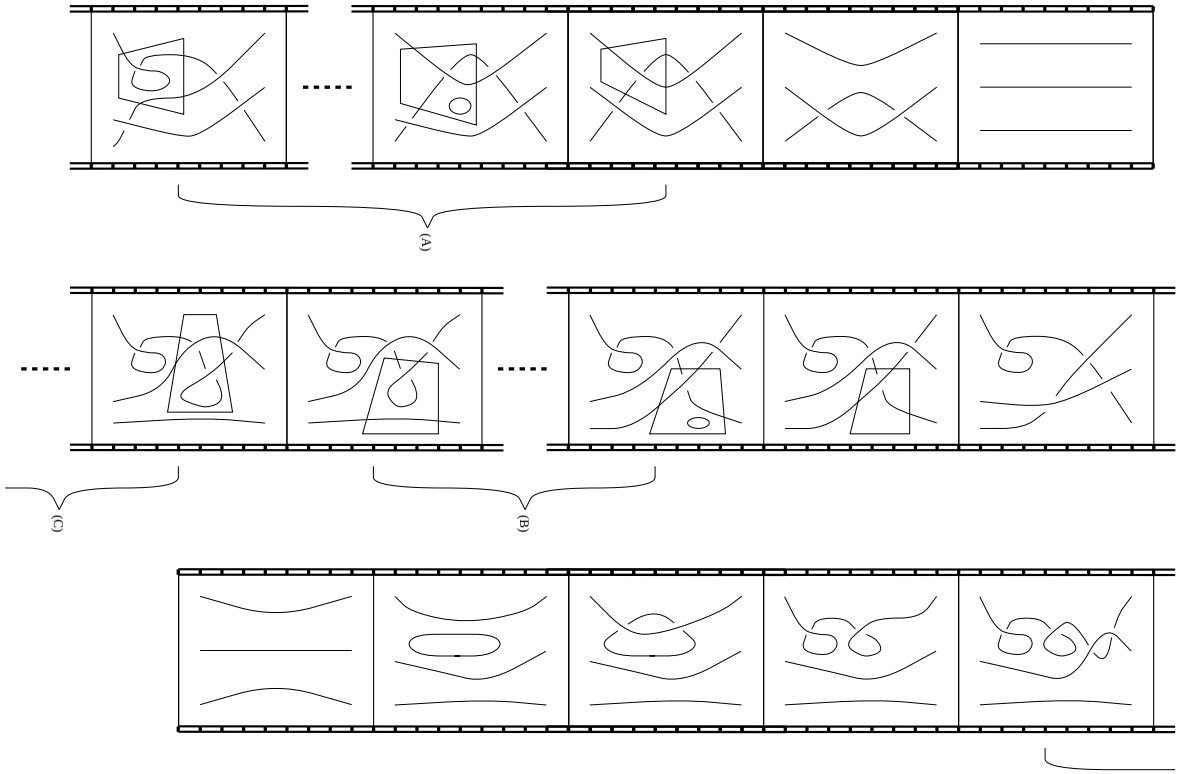
(A)
(C)
(B)