This story begins from powers of 2; starting from $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$, and $2^{10} = 1024$ are the first few values.

The other day I received a chain letter by email that asked me to mail a copy to 10 people whom I know. I patiently explained to the sender (whom, I did not know), that exponential growth is a dangerous thing. within 10 generations (and that send button is easy to hit) everyone in the world would have received a copy of the letter.

The standard myth is the emperor of China asking the inventor of chess the price of the game: one grain of rice for the first square, 2 for the second, 4 for the third, and double the number of grains of rice for each successive squares. By the end of the board the emperor owed

$$1 + 2 + 4 + \cdots + 2^{64} = 2^{65} - 1$$

grains of rice. Now for a ball-park estimate of this number, remember that $2^{10} = 1024$. So $10^3 = 1000 < 2^{10}$. Therefore,

$$2^{65} = 2^5 \cdot (2^{10})^6 > 32 \times (10^3)^6 = 32 \times 10^{18}$$

Or

$$2^{65} - 1 > 32,000,000,000,000,000,000.$$

The national debt of the US (the largest negative number of which I can conceive) has absolute value $\$8,684,638,939,701.35 > \$10^{13}$. The price of a 25 pound bag of rice is about \$10. My very rough estimate is that a pound of rice has about 5,000 grains. So the price per grain of rice is something like \$1/12,500. So the emperor of china owes the chess board inventor about $\$3 \times 10^{15}$. Chess is an ancient game, and I presume that there should be interest charged on the debt to the inventor. This too, is an exponential growth problem, but I digress.

There are many things besides grains of rice distributed that powers of 2 count, and I want to describe some of them. Certainly the whole numbers from 0 to $2^n - 1$ can be counted by 2^n. And we have a nice way of doing this. We express a number in its binary notation. Here is a table to help you understand this idea:

n	Binary representation
0	$(000)_2$
1	$(001)_2$
2	$(010)_2$
3	$(011)_2$
4	$(100)_2$
5	$(101)_2$
6	$(110)_2$
7	$(111)_2$

The digits in 6, for example. indicate that $6 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$. And I have padded the initial portions with 0 so that we can see that these powers don't appear.

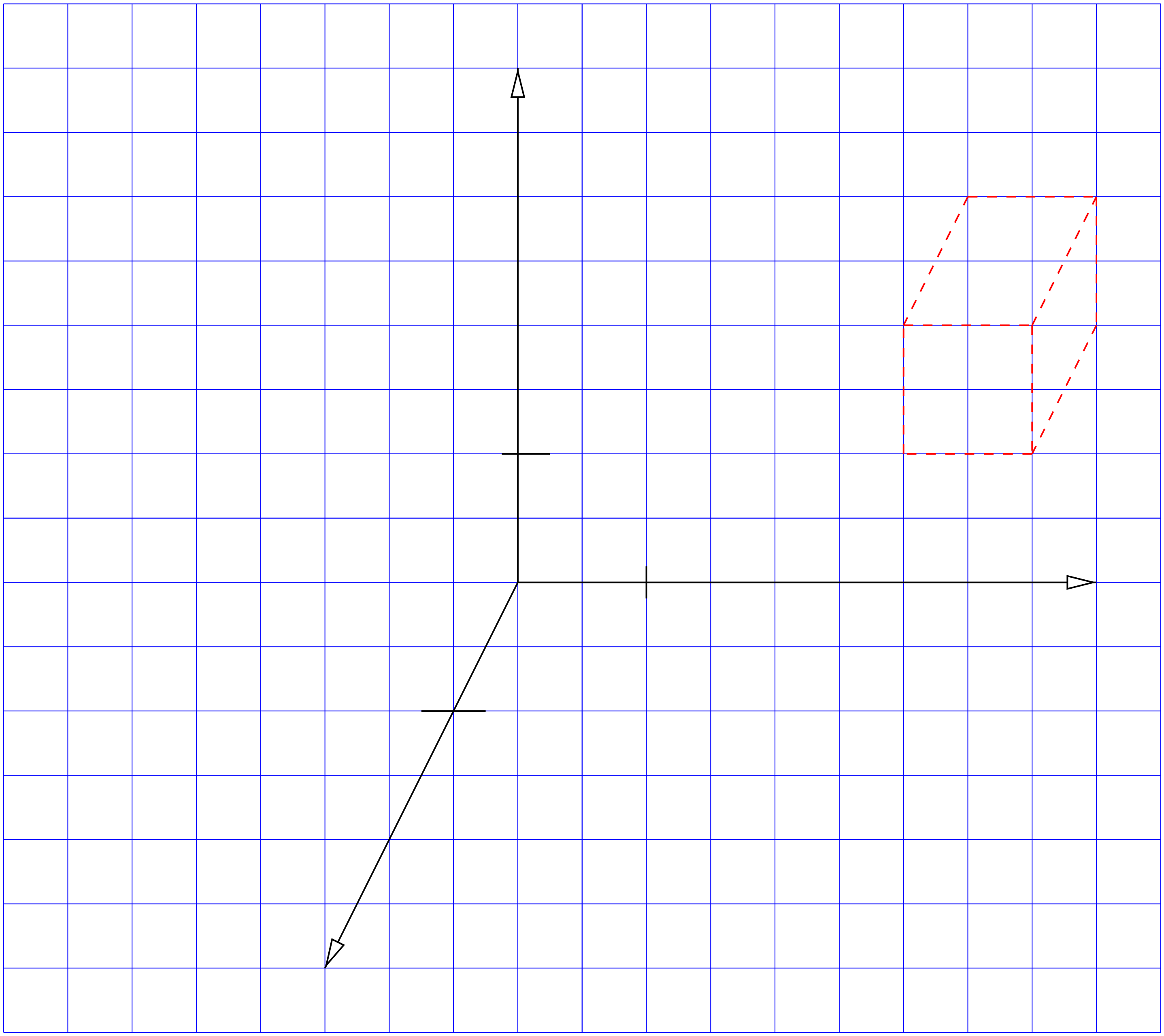
The binary representation of a number indicates which powers of 2 to include to express the number. So the number 6 requires a 4 and a 2. The number 100 requires one 64, one 32, and one 4. So $100 = (1100100)_2$.

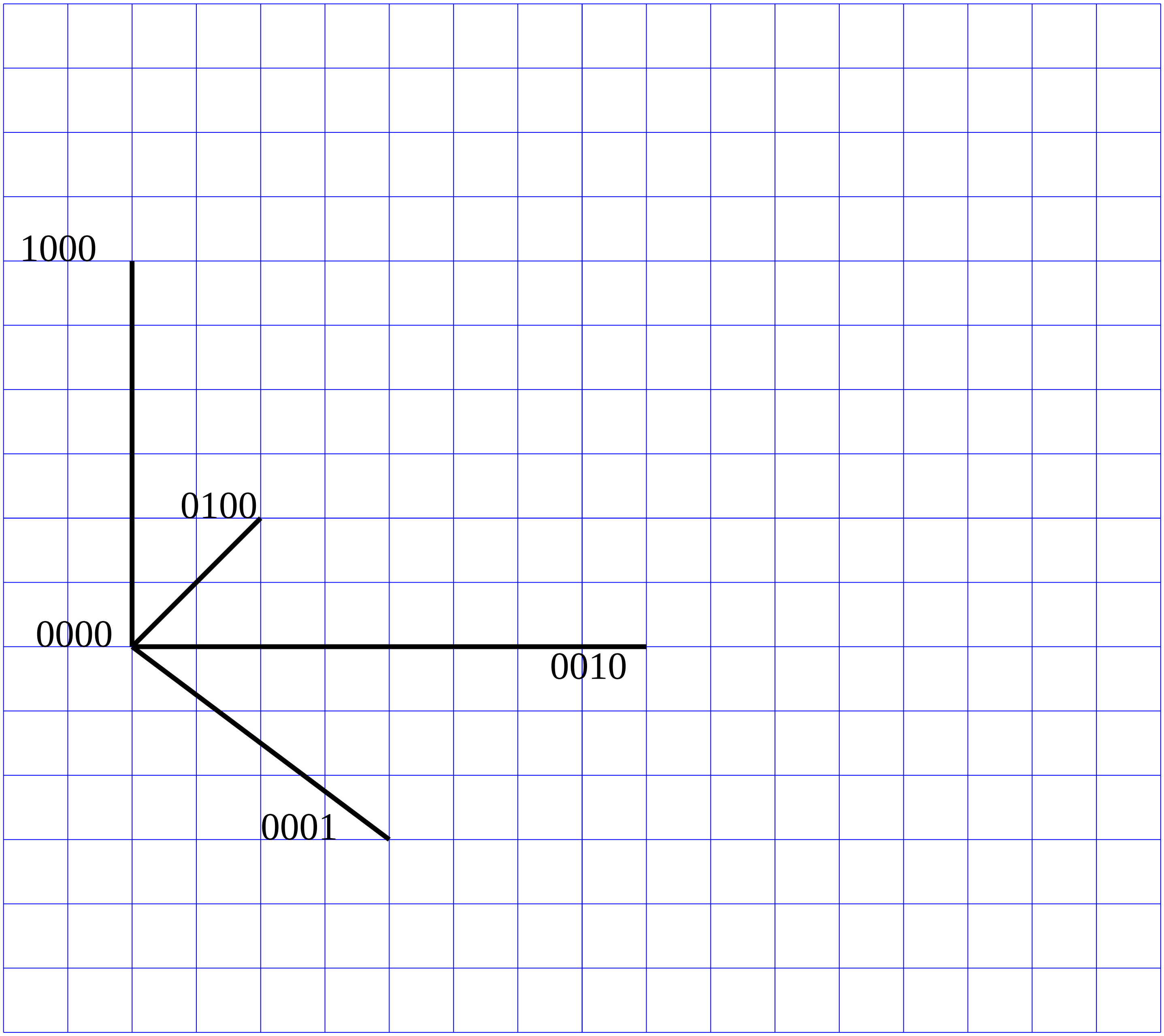
Another thing that is counted by powers of 2 is the collection of subsets of a set of fixed size. For example, if we have a 2 element set, $\{1, 2\}$, then we have the following four subsets, $\{\}$ — the set with no elements, $\{1\}, \{2\}$ — the two sets with one element each, and the two element set $\{1, 2\}$. For a three element set, we have the following eight subsets, $\{\} \quad \{1\}, \{2\}, \{3\}, \quad \{1, 2\}$,$\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. Of course a set with one element has two subsets: $\{\} \quad \{1\}$. And the empty set has only one subset.

We can group subsets according to their size, and we have the following sequences,(1) $(1, 1)$, $(1, 2, 1)$, $(1, 3, 3, 1)$, $(1, 4, 6, 4, 1)$. Those in the know, recognize this as the sequence of terms in Pascal's triangle. Traditionally, we think of Pascal's triangle as determining the coeficients of of $x^i y^j$ in the expansion of $(x + y)^n$.

Now why does two to a power count the number of subsets of a given set? Why can these subsets be arranged in size according to Pascal's triangle? Is there a convenient way of arranging these subsets — to keep track of them and relationships among them?

To answer these questions, I will begin in the most simple cases, and I will gradually build up the dimension ladder.

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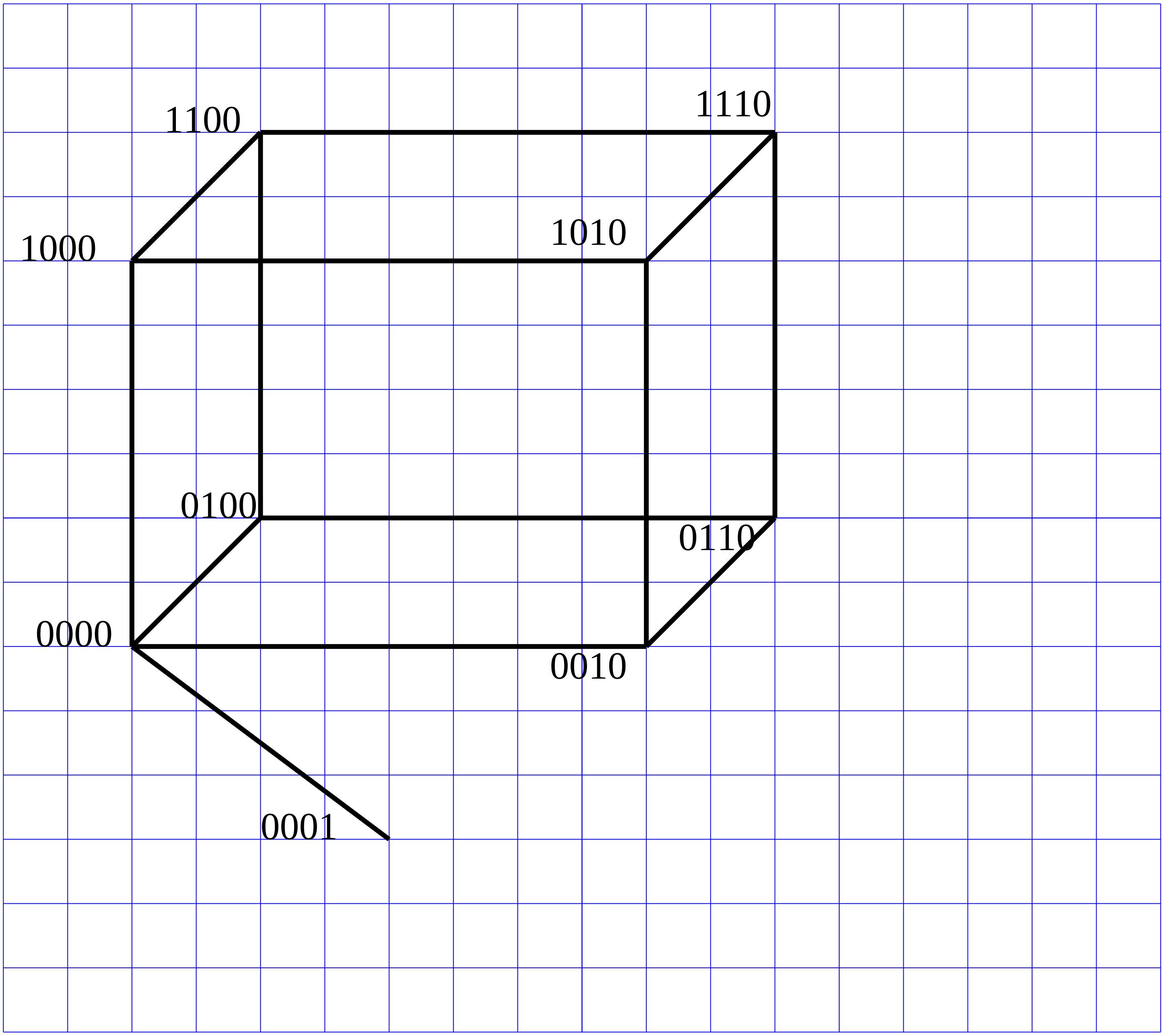
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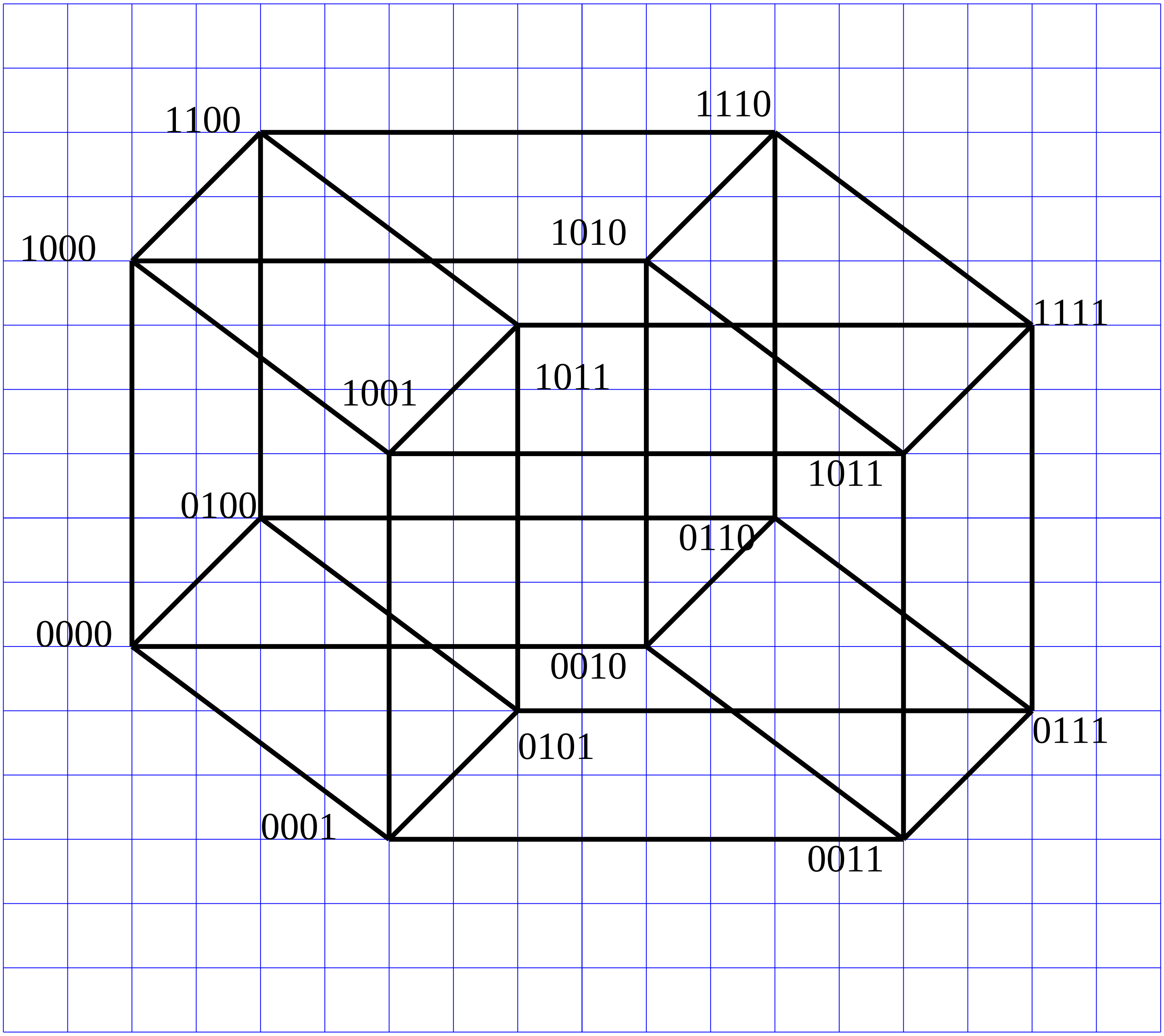
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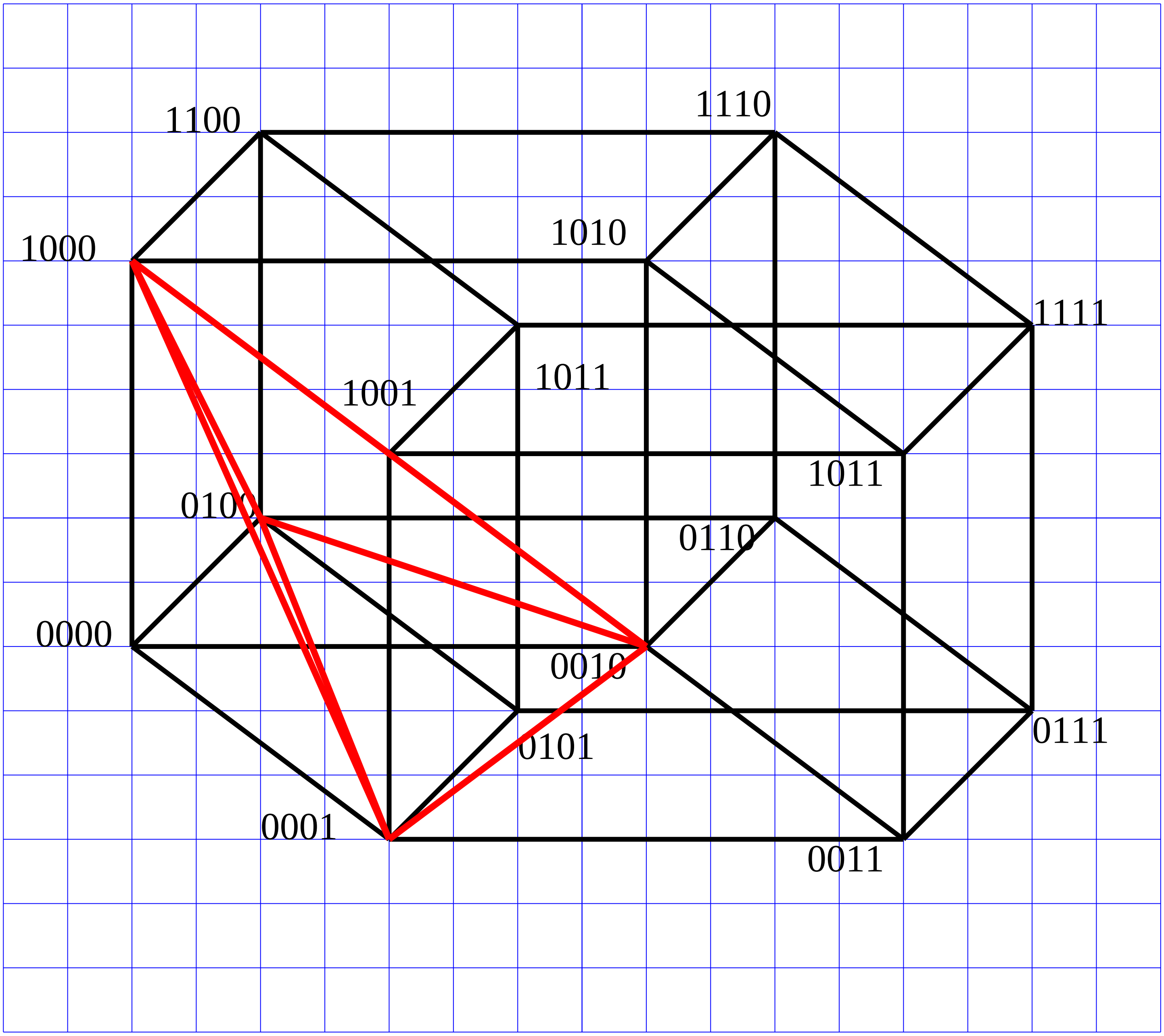
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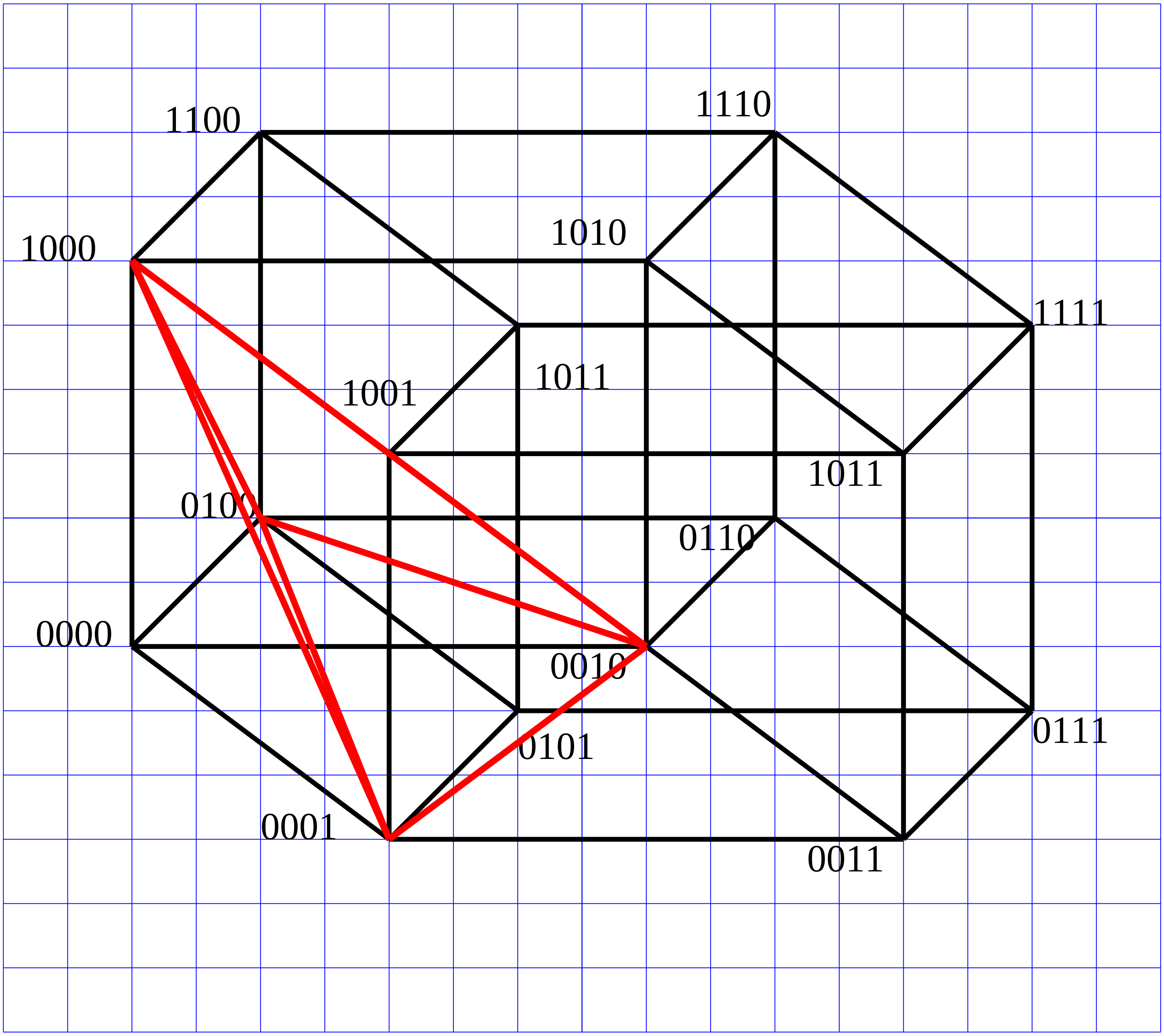
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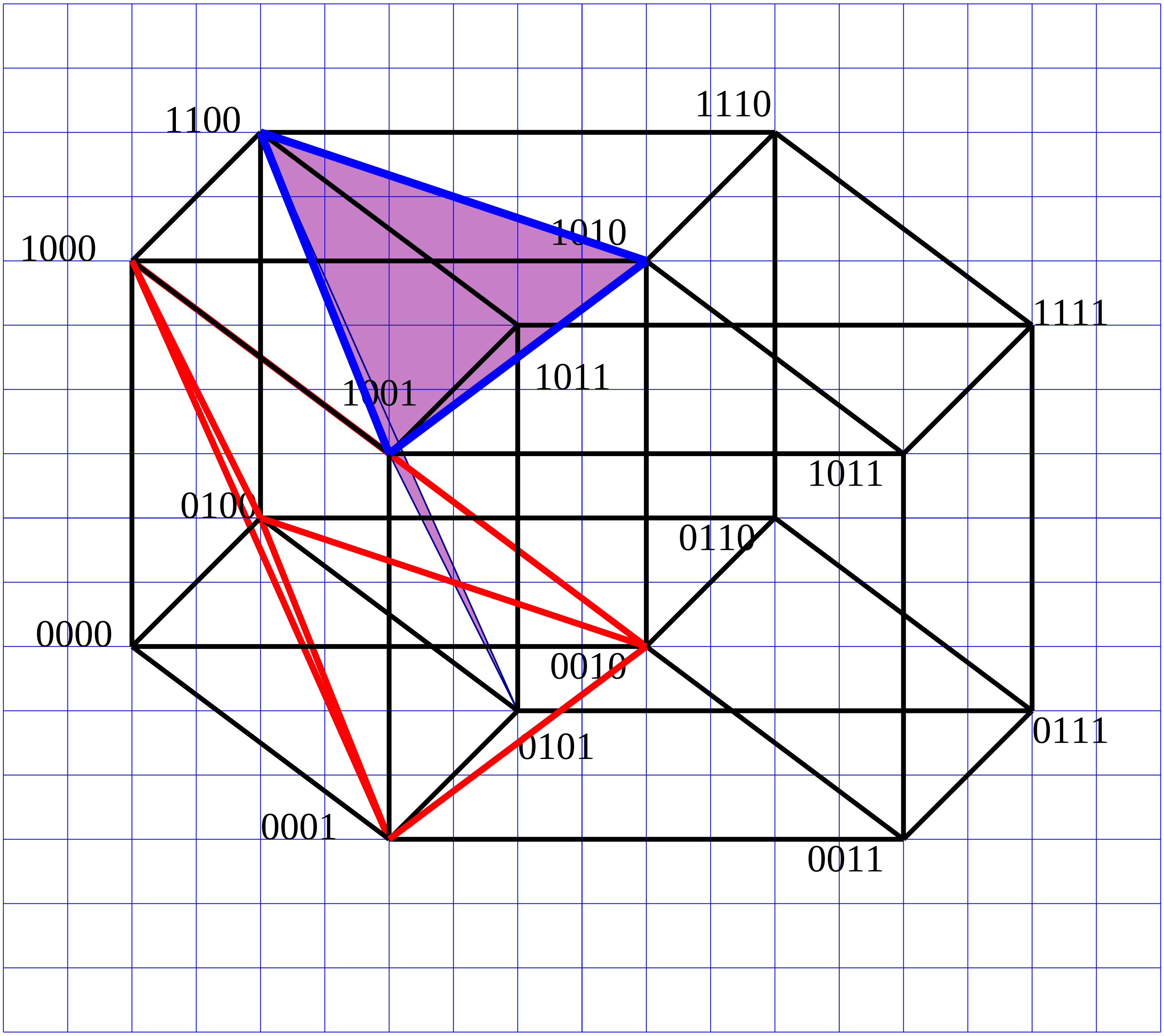
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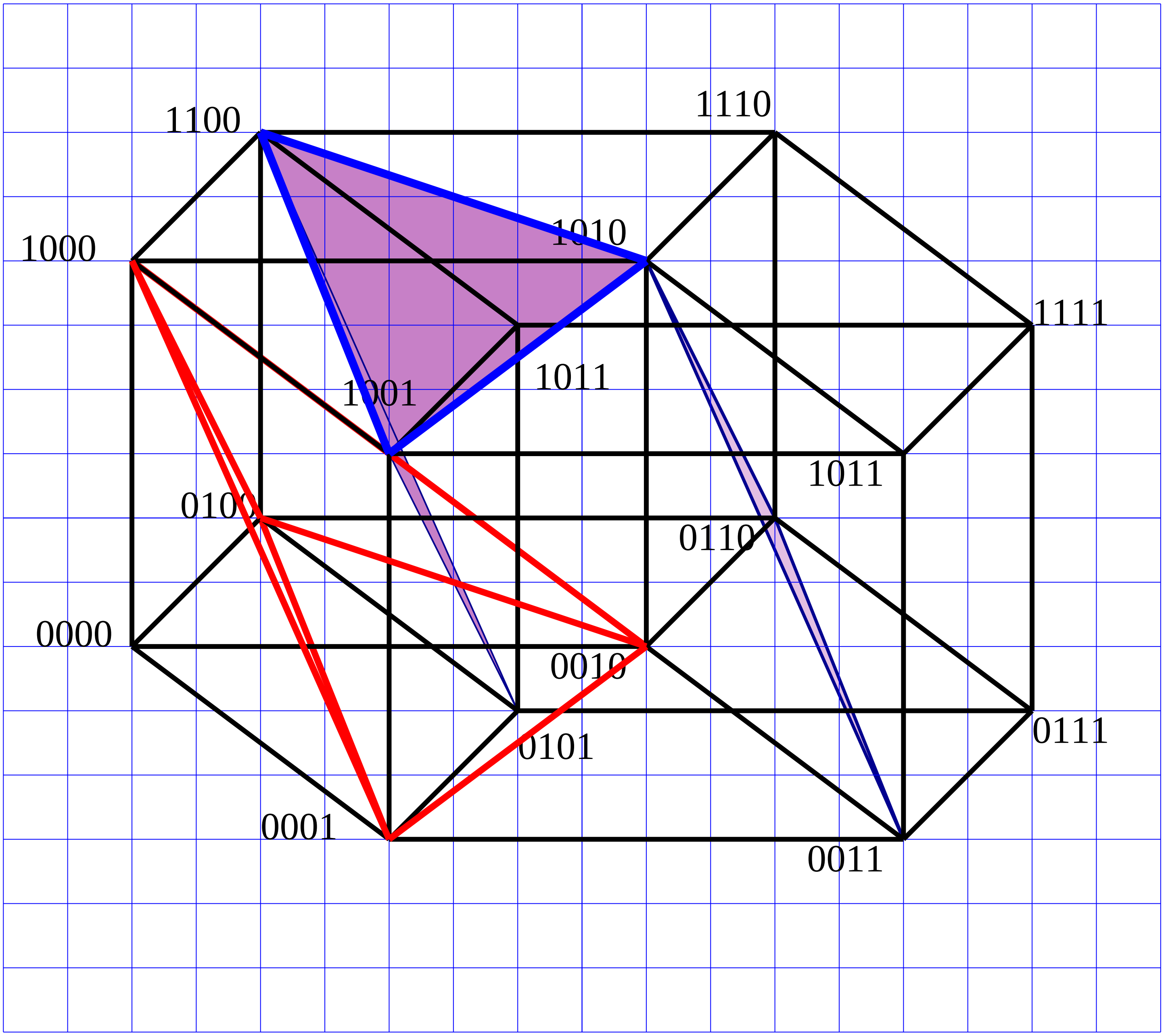

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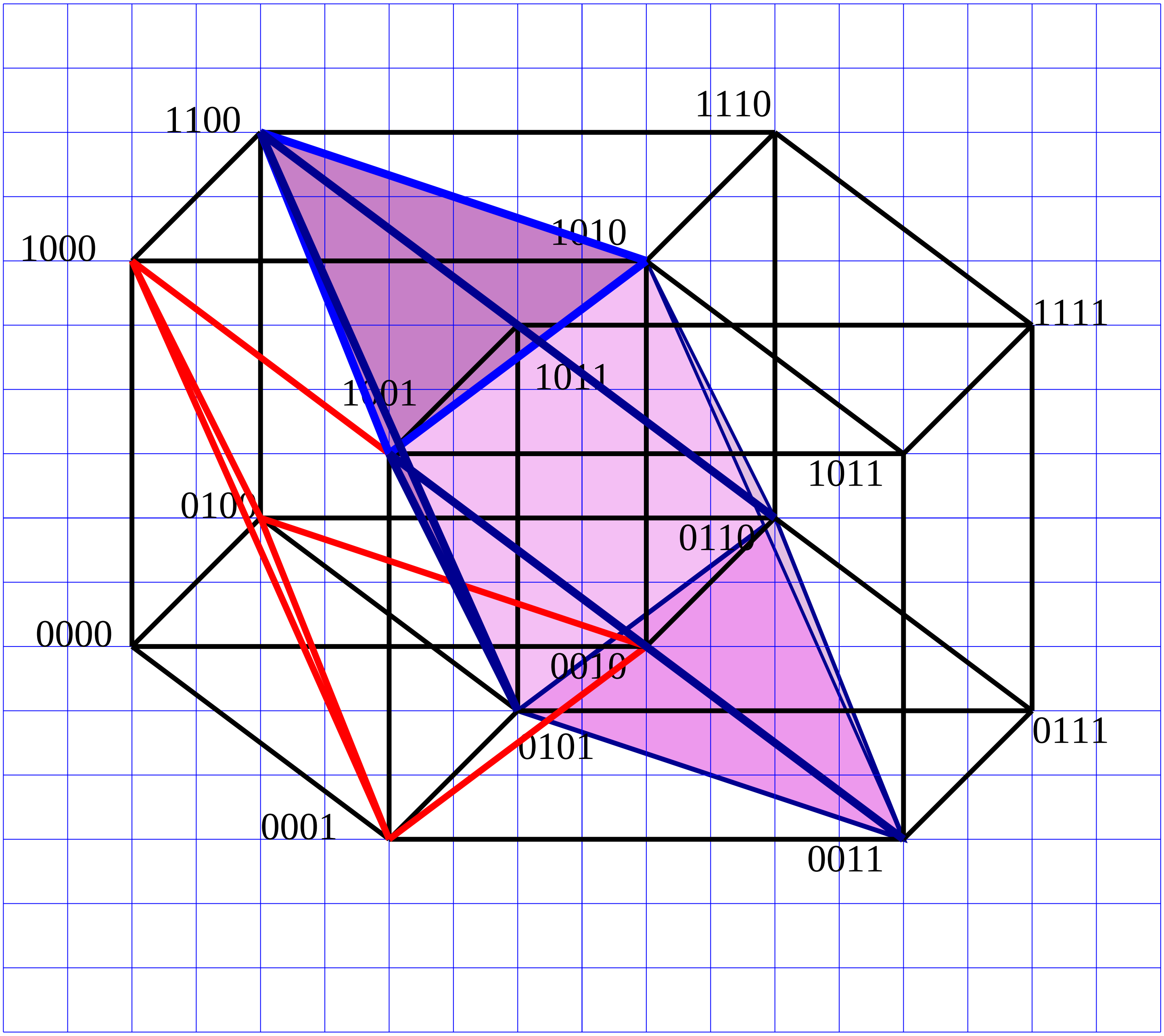
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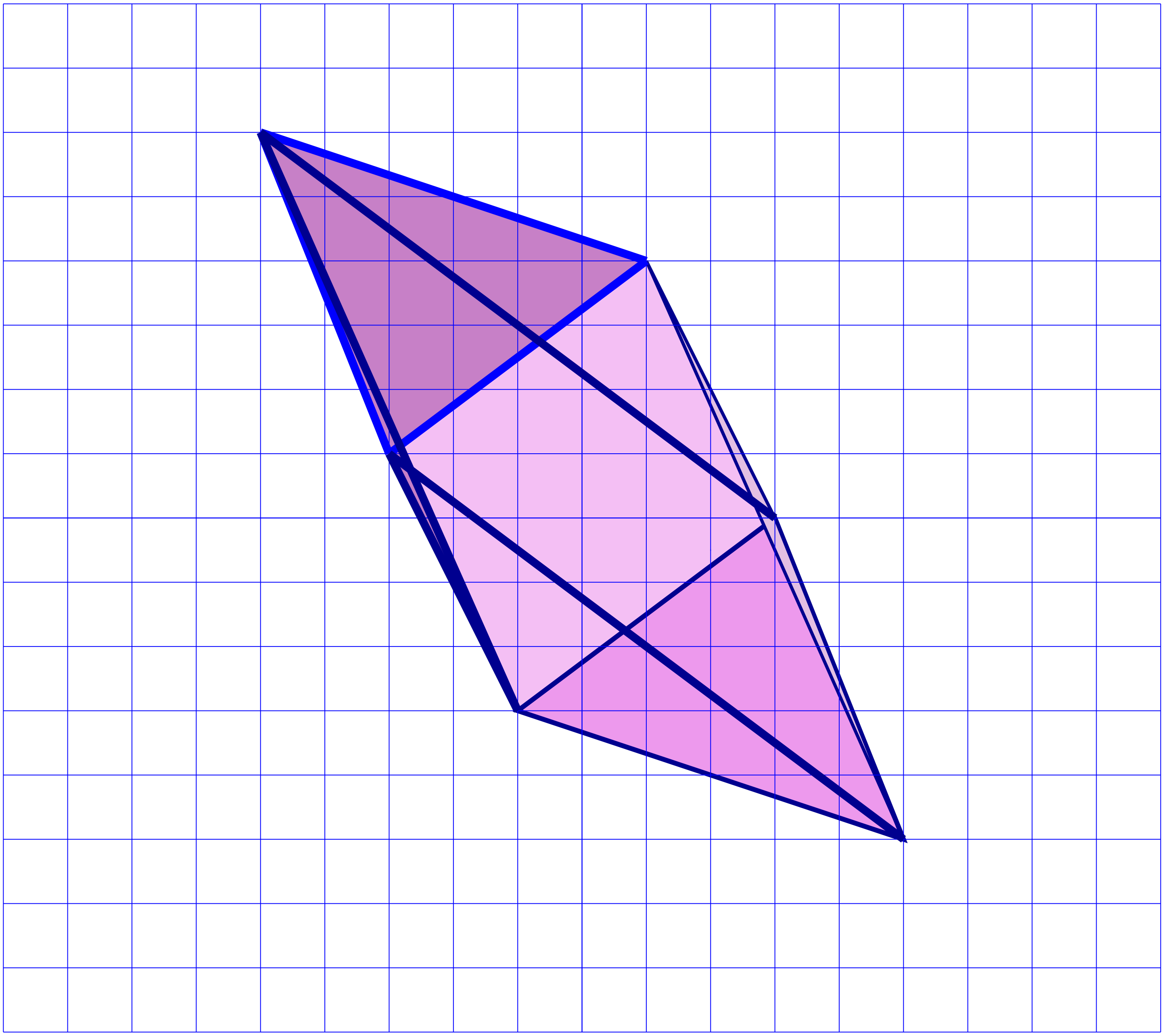

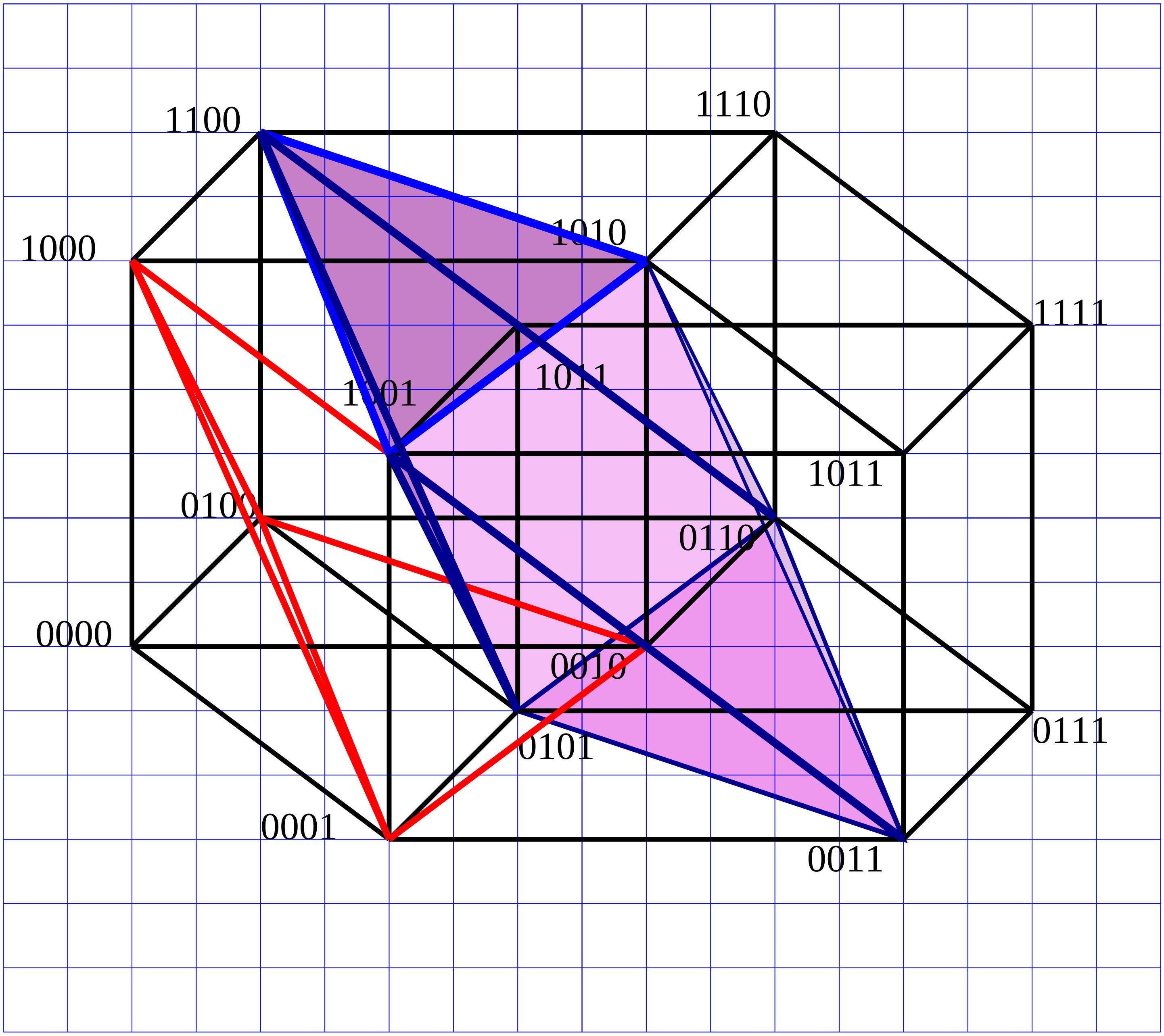
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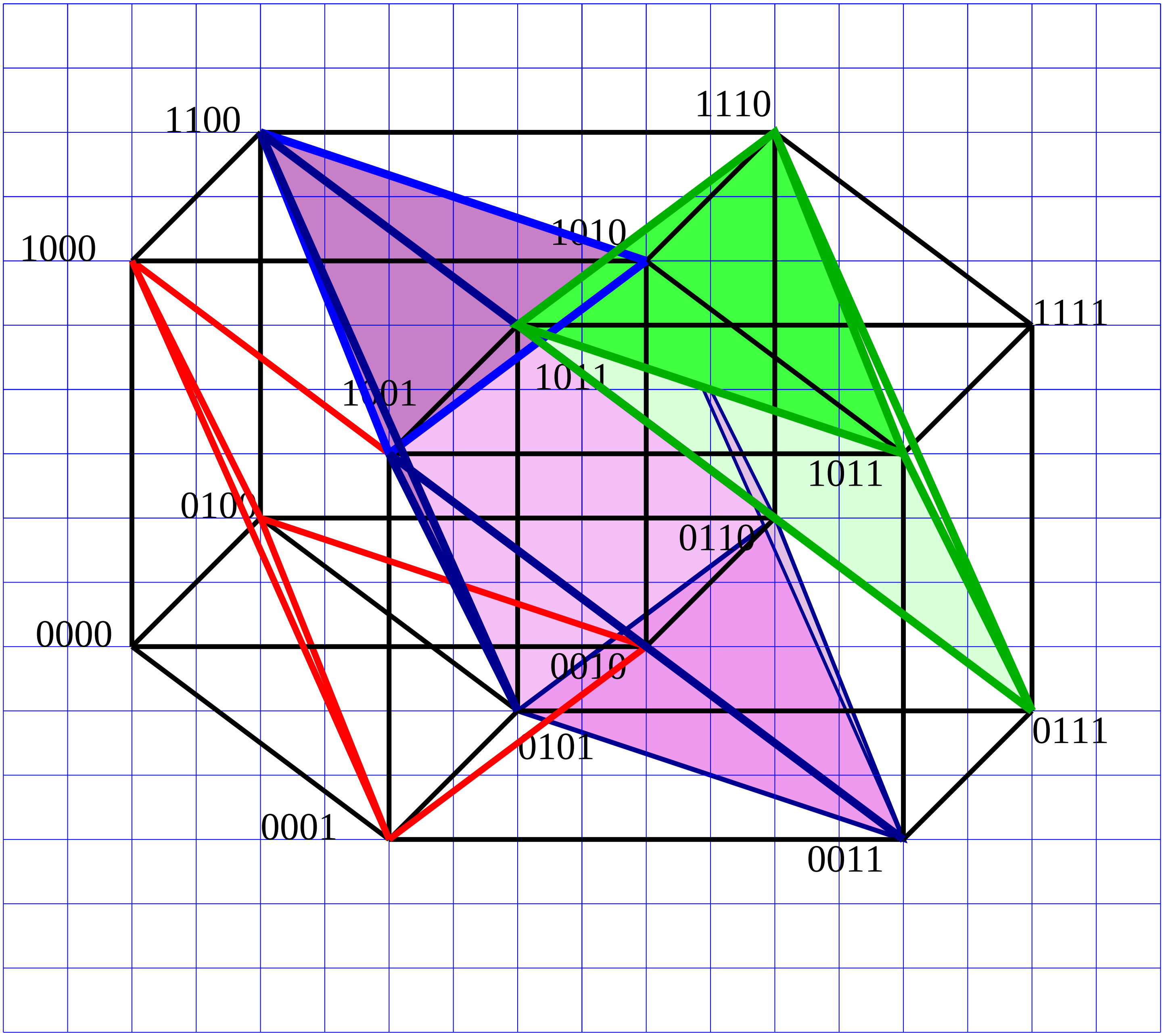
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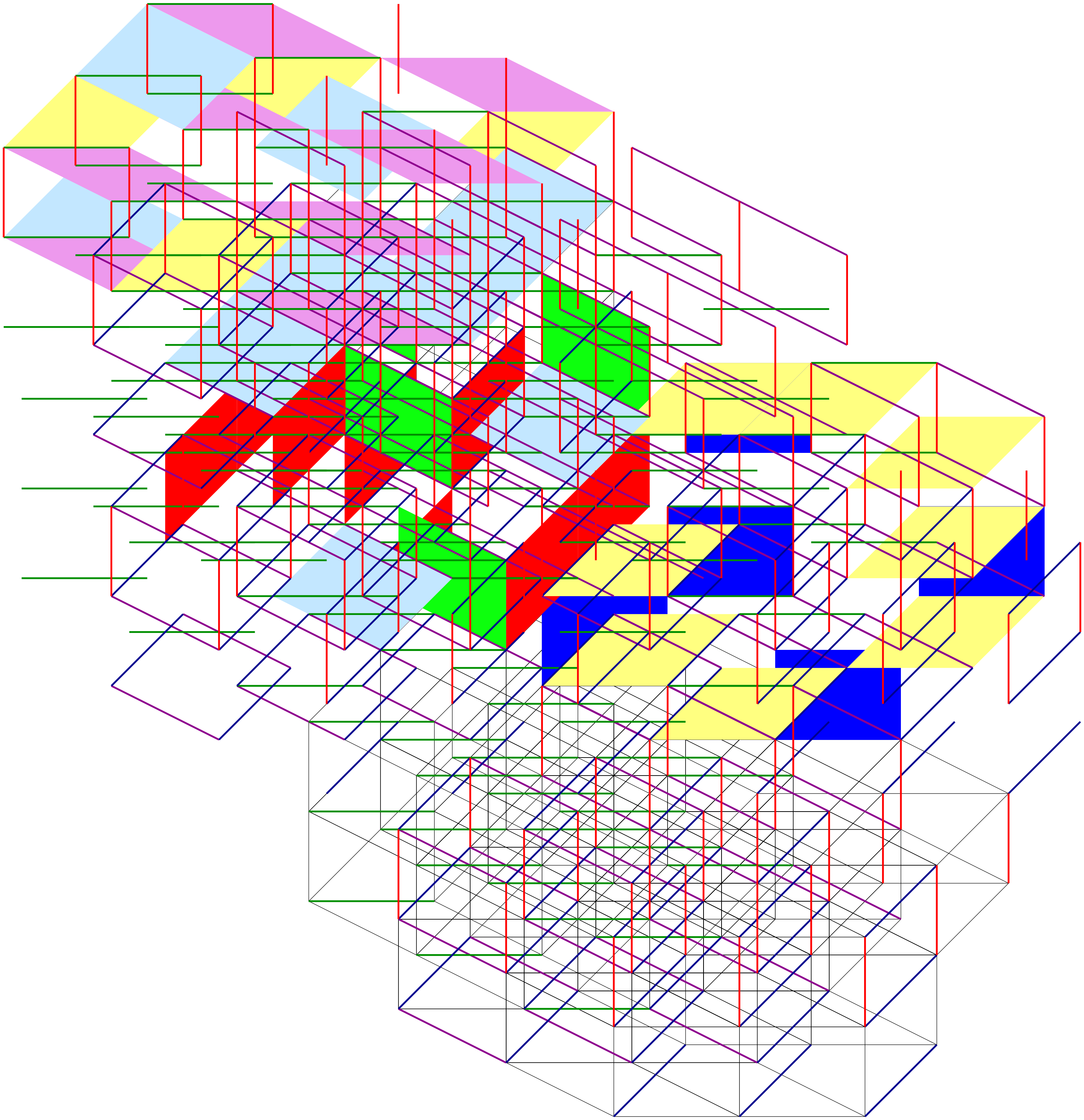

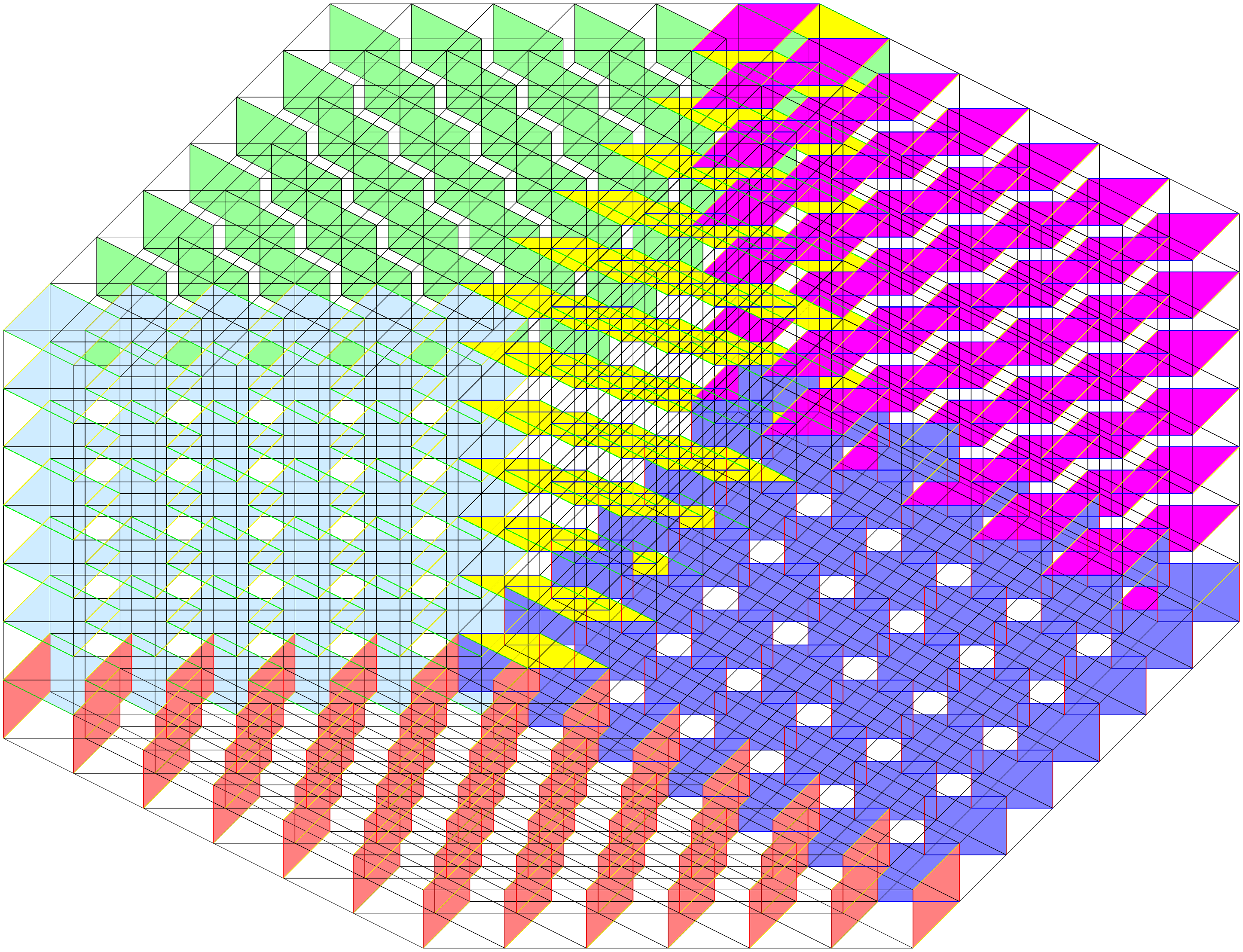

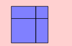

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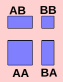

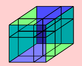

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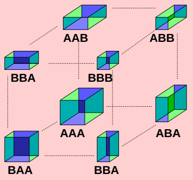

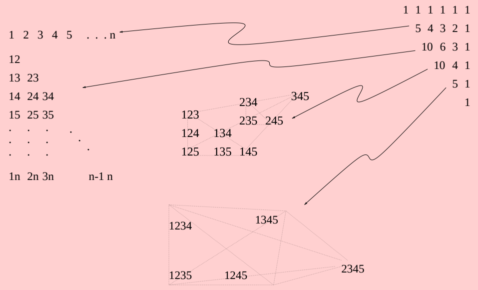

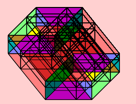

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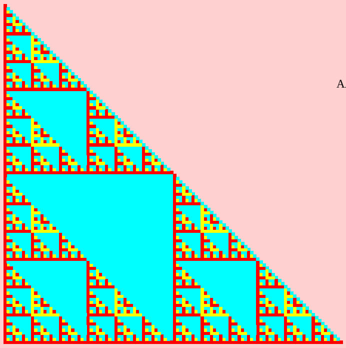
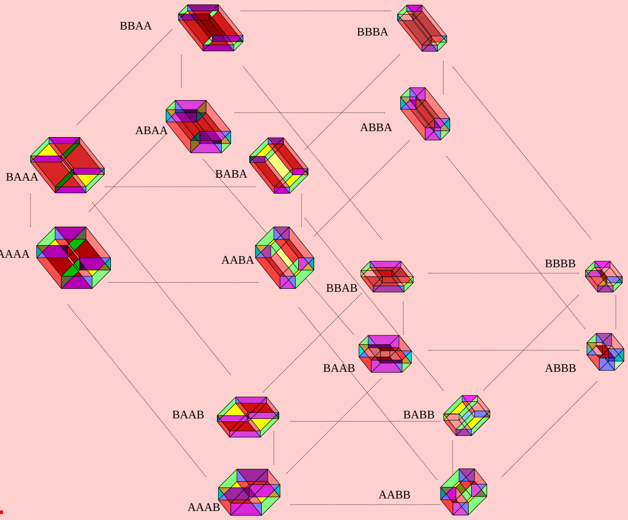
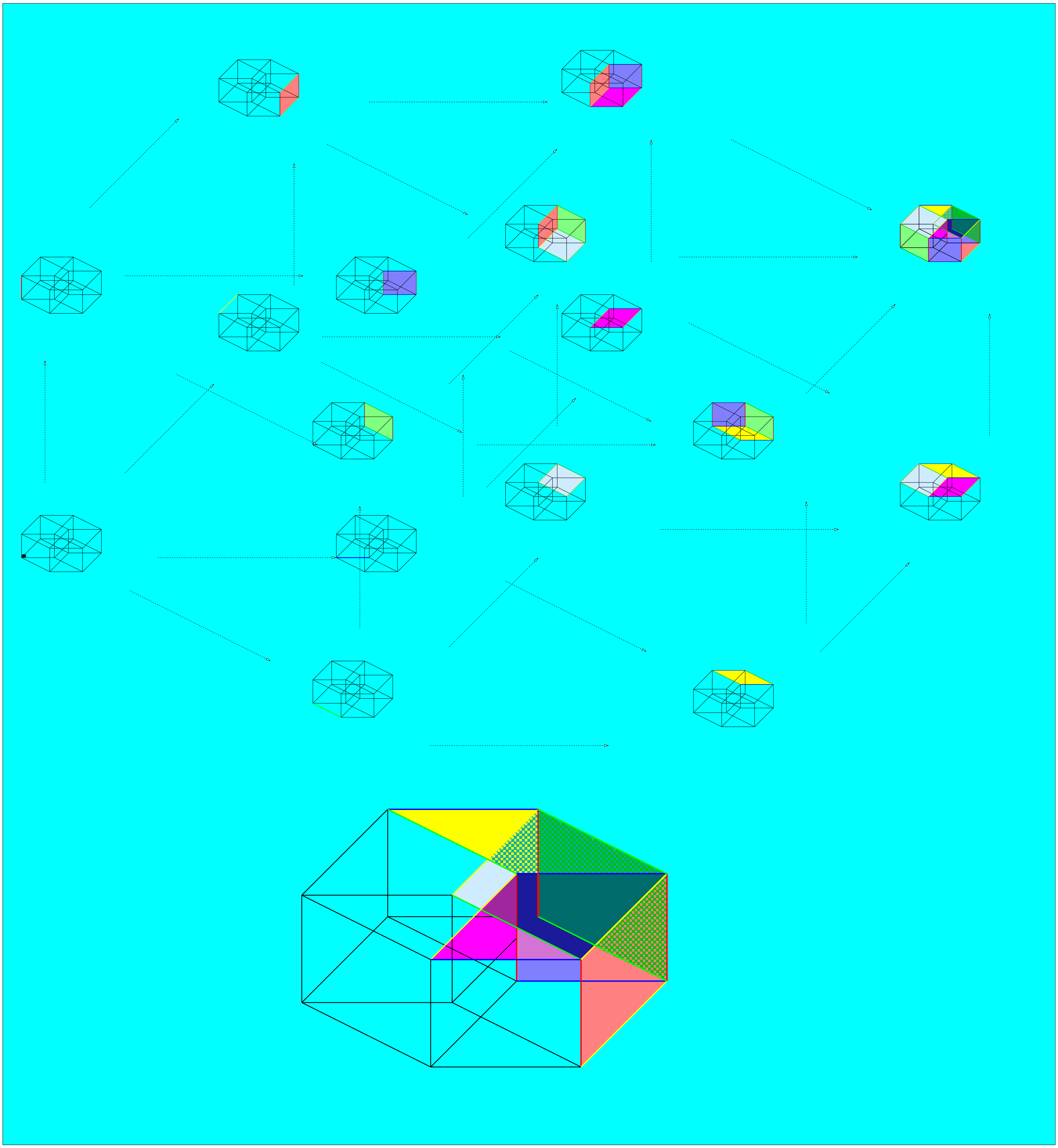
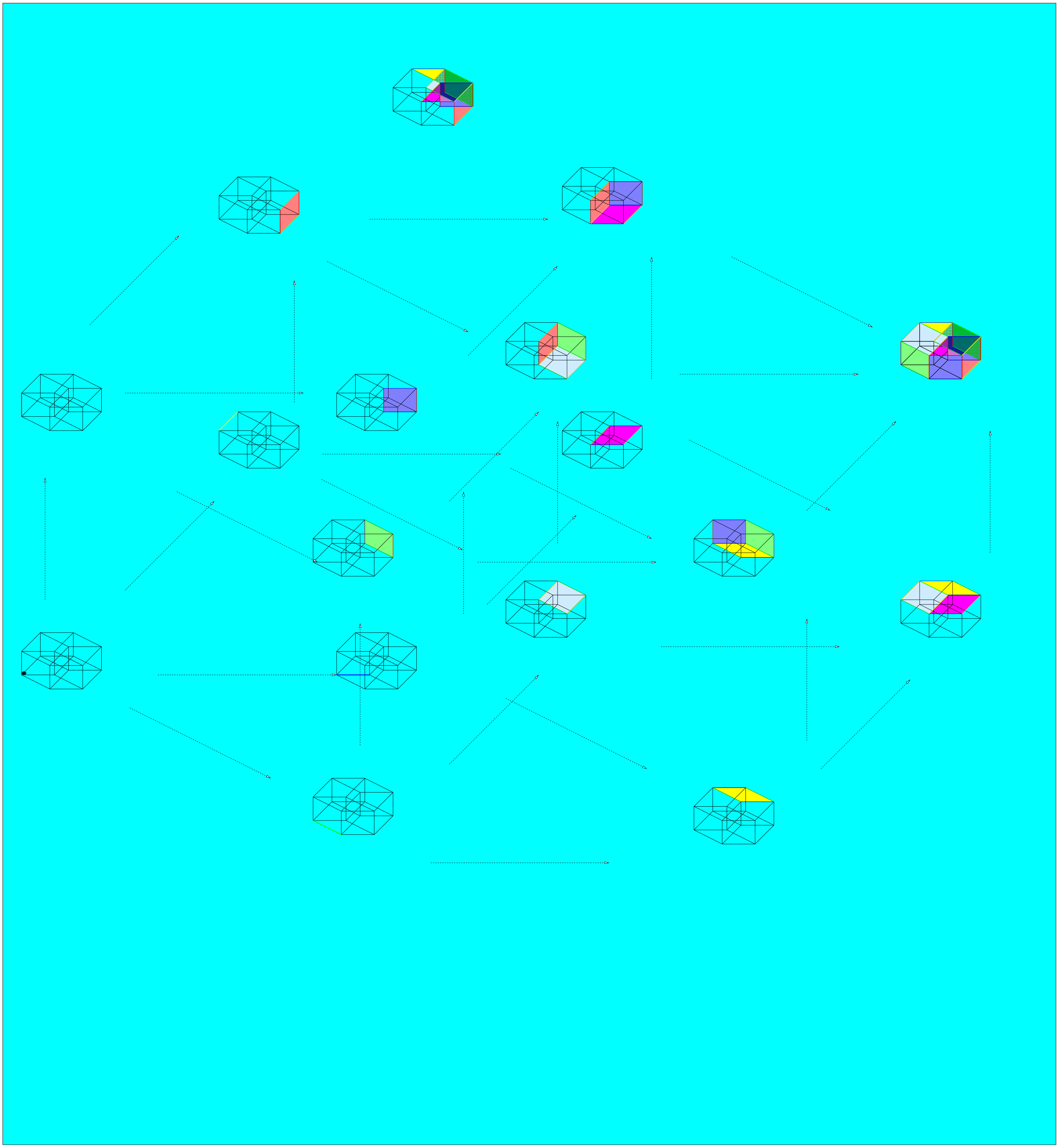


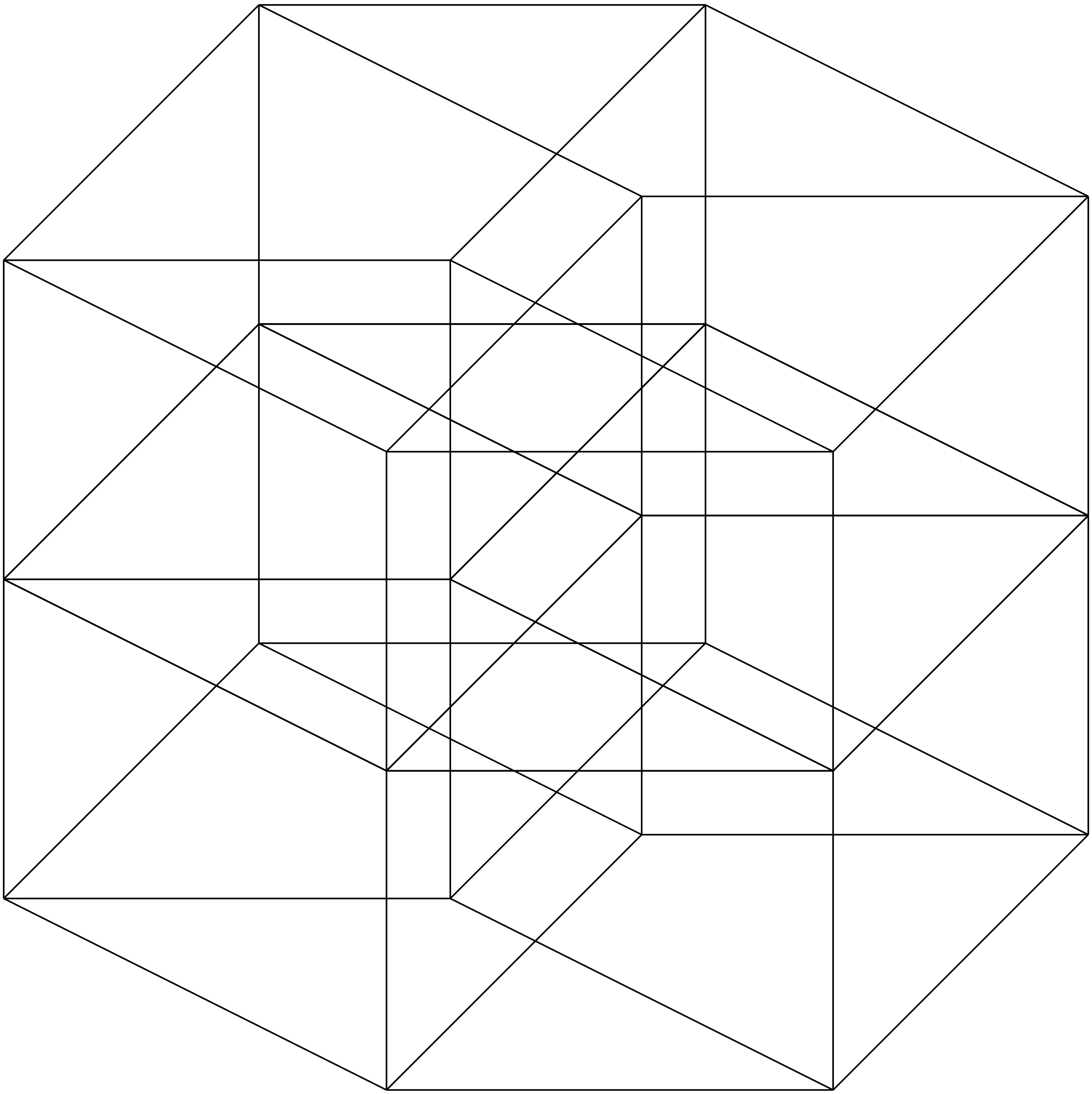


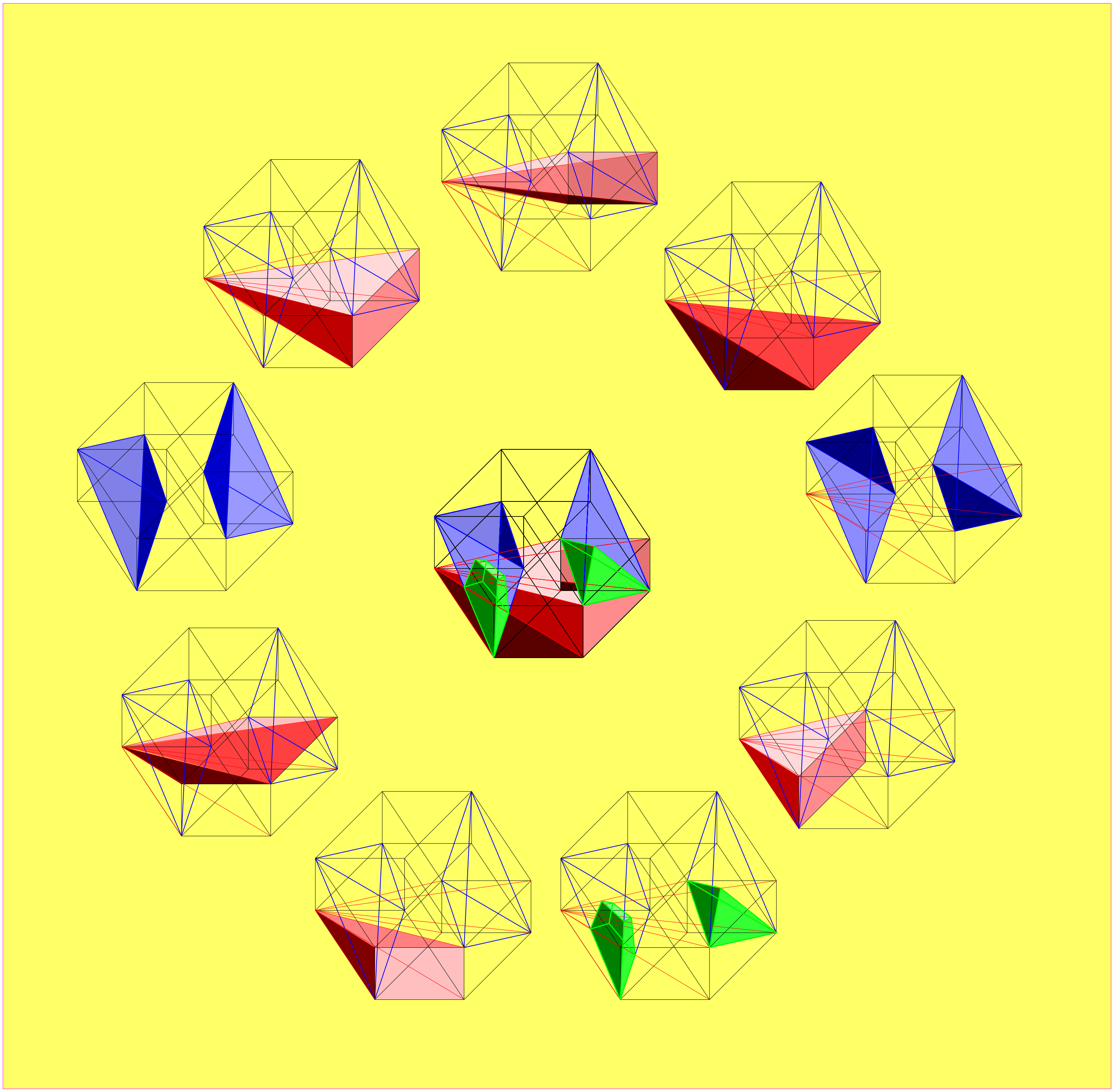
The Binomial Theorem

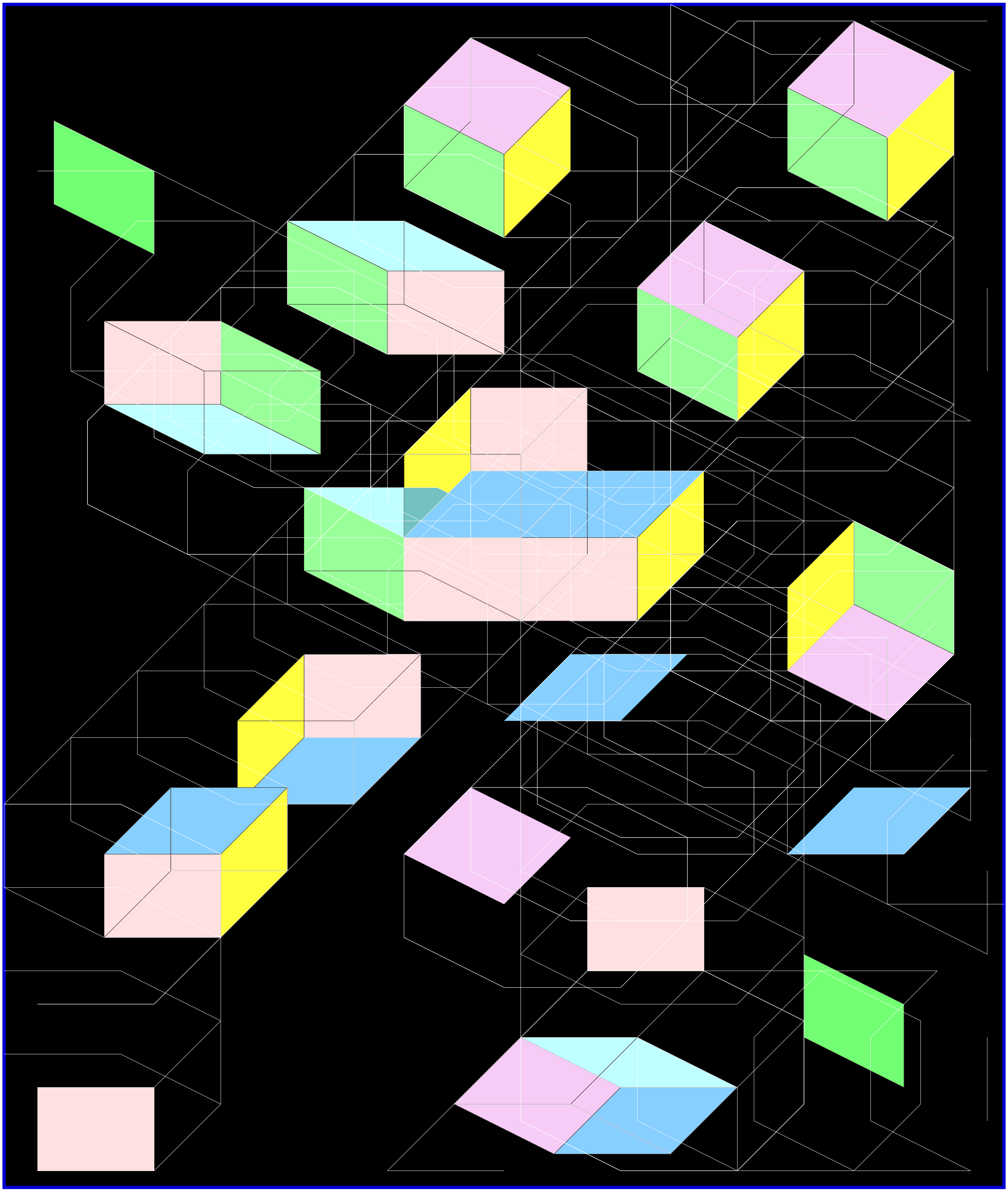
expresses the nth power of the sum, A+B, as the sum over all binary sequences of As and Bs. The power of the sum is the hyper-volume of the hypercube whose edge length is A+B. The binary sequence summands are each the hyper-volume of a hypercube whose edge lengths are some choice of As and Bs. Pascal's triangle tells us how many terms of each type.

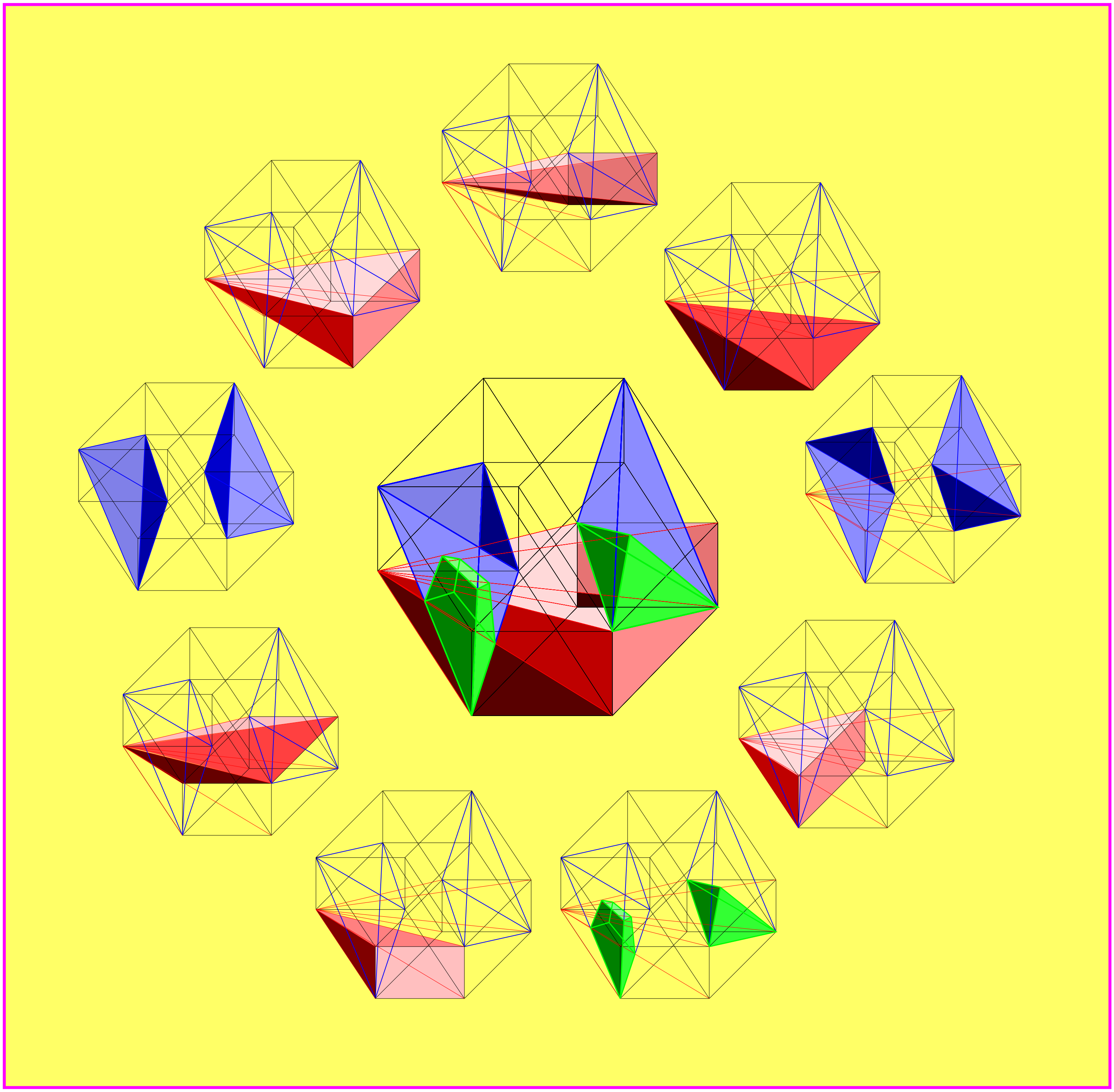


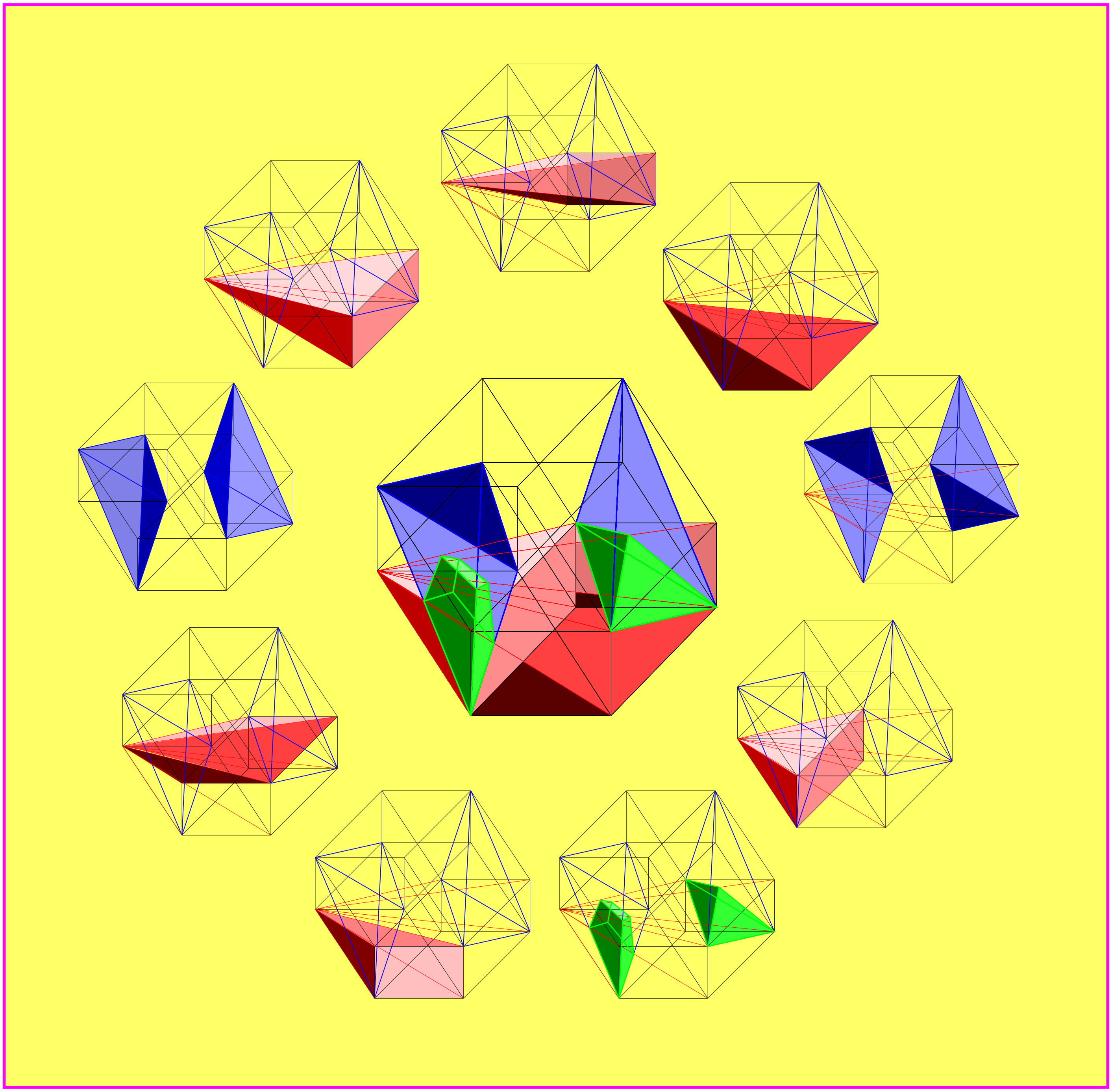




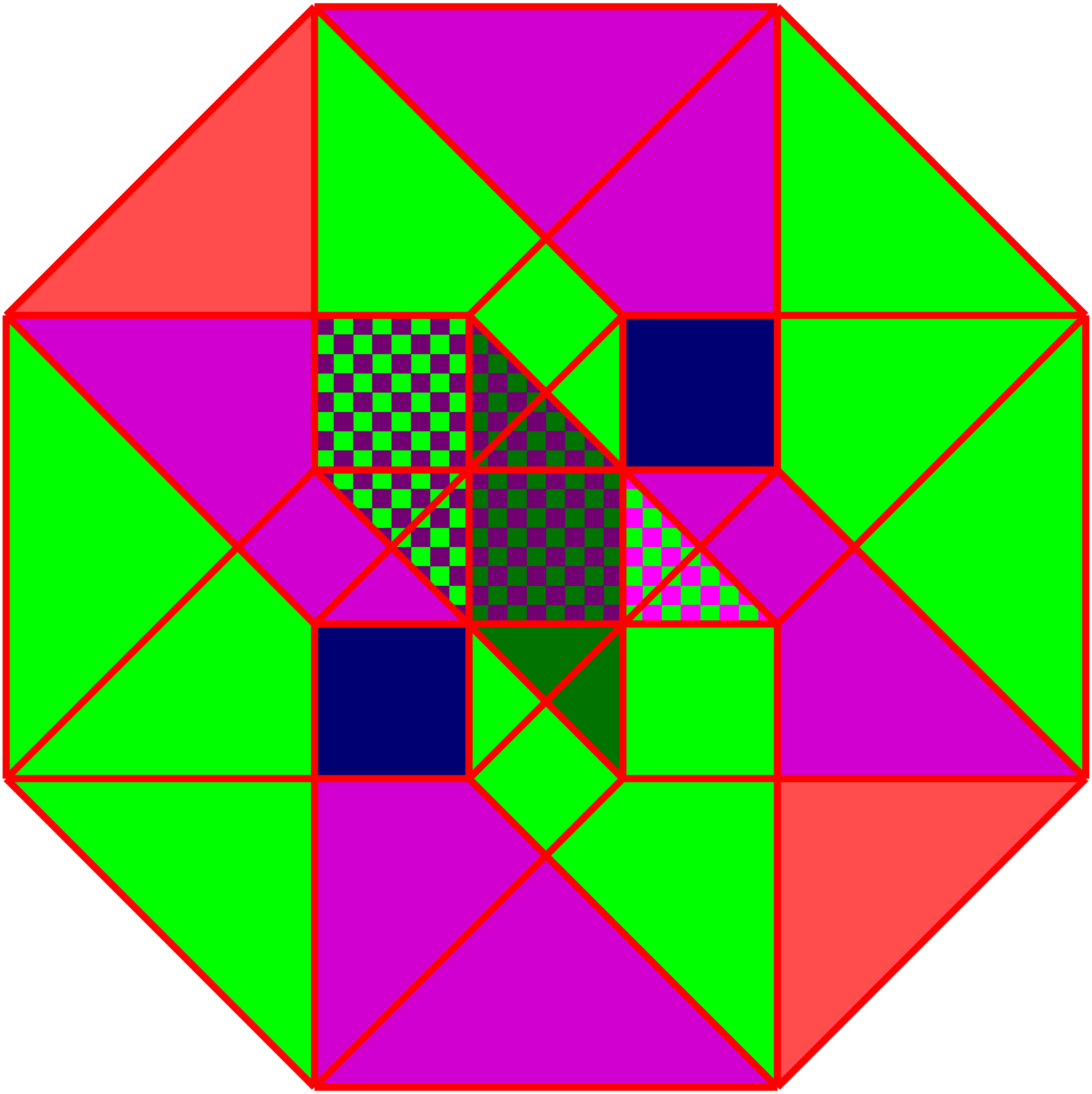


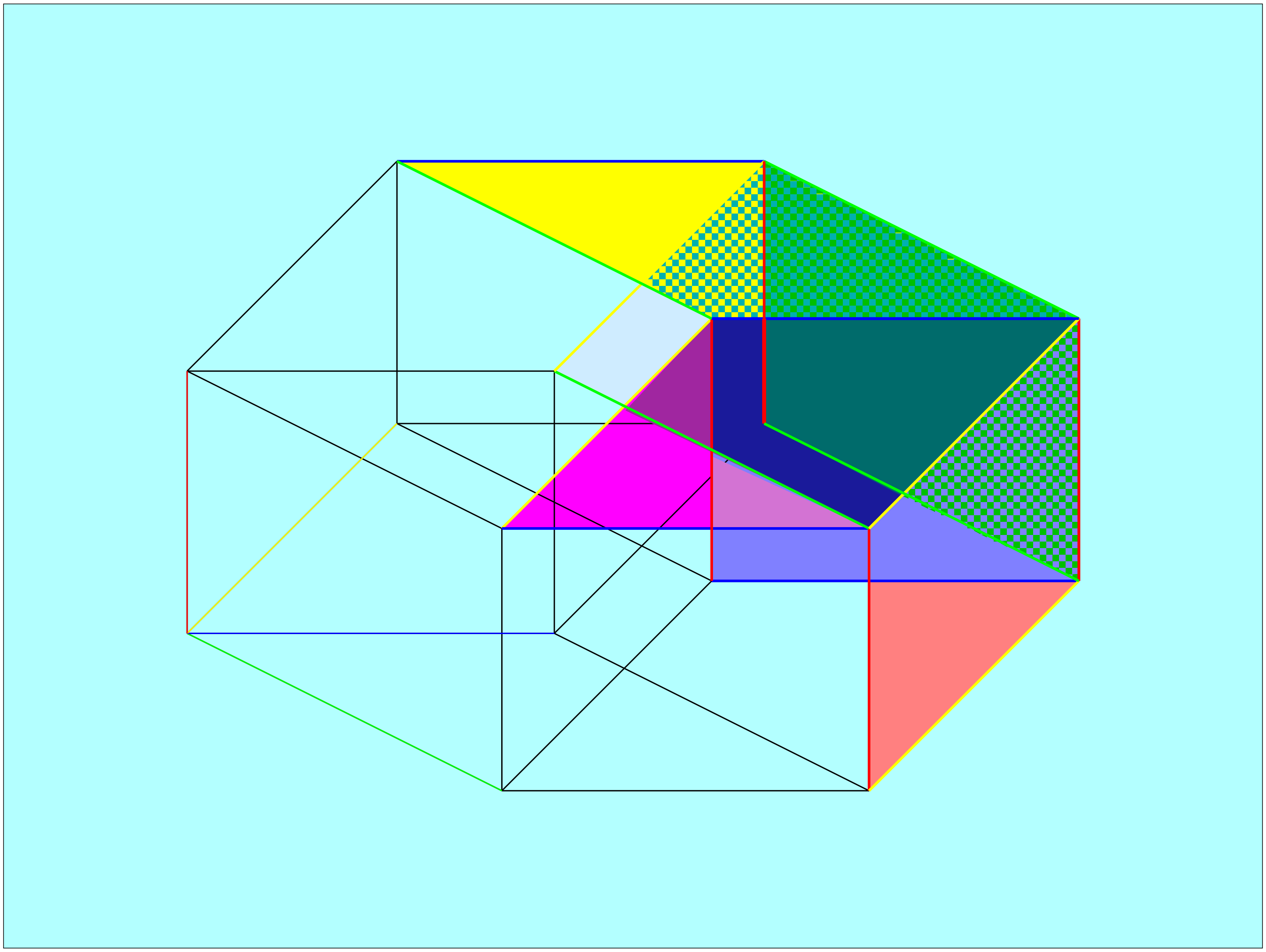


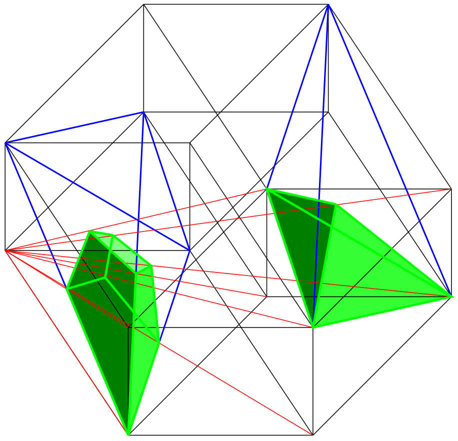


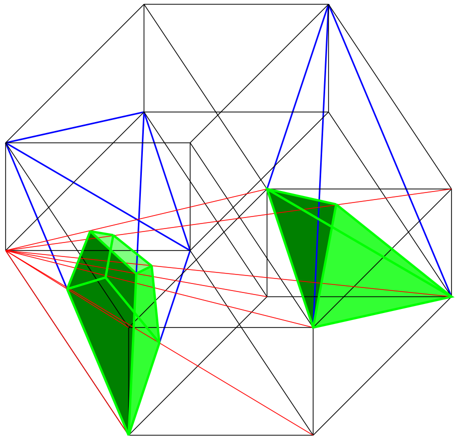


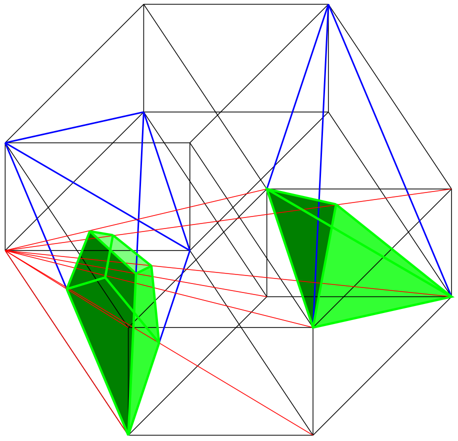
Inside a 4-dimensional cube, the 3-dimensional top cell (x,y,z,1) is coned to the origin, (0,0,0,0). The red pyramids indicate the six pyramidal cells on the various 2-dimensional faces of that cube. The blue tetrahedron on the left is the smallest convex set that contains the unit coordinate vectors. The The blue tetrahedron on the right is the smallest convex set that contains the four points (1,1,1,0), (1,1,0,1), (1,0,1,1), and (0,1,1,1). The red cone intersects these two tetrahedra in green solids. On the left is an object that has 2-dimensional faces which are kites and which together form a cube-like object. On the right, four triangular faces form a tetrahedron. Either of these figures can be rotated and reflected within the 4-cube, so that four copies of the green objects tile the blue tetrahedron. Under such rigid motions, the cubical base of the red pyramid rotates to (x,y,1,w) then to (x,1,z,w) and finally to (1,y,z,w). Meanwhile, the resulting four copies of the 4-dimensional pyramid fill the 4-cube. Thus the original cone occupies a quarter of the space in the 4-cube. Therefore, the sum of the volumes of cubes whose edge lengths range from 0 to 1 is 1/4. Analogous phenomena occur in all dimensions.

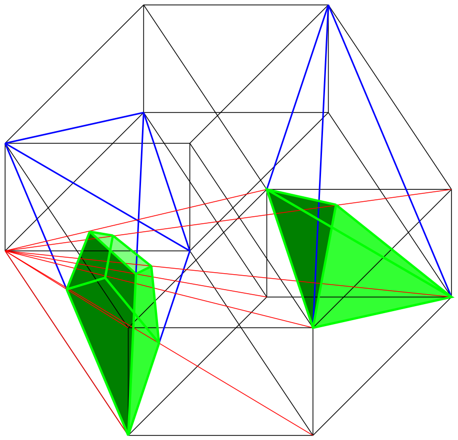


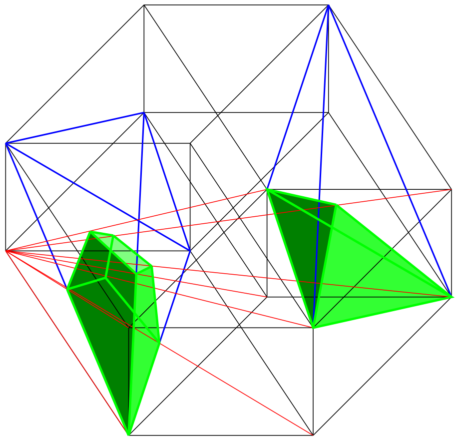


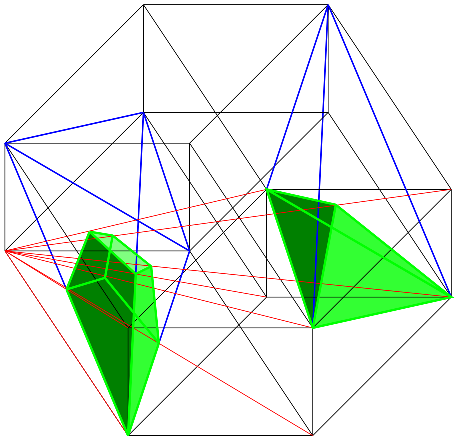


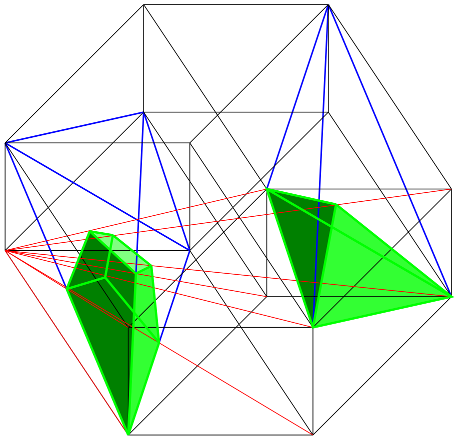


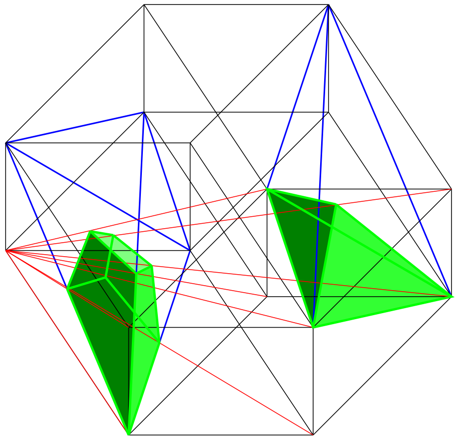


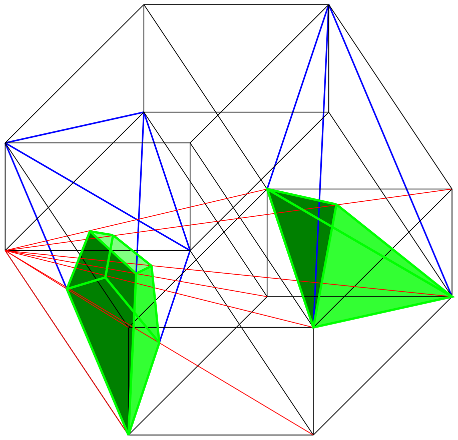


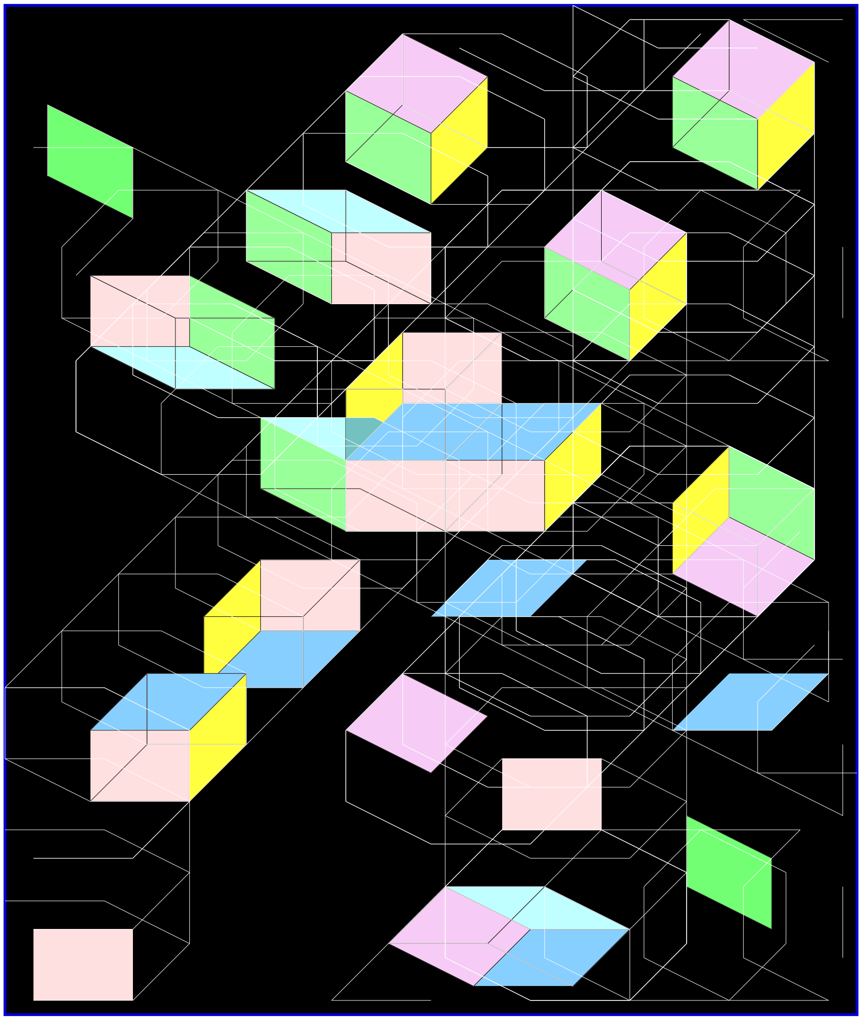


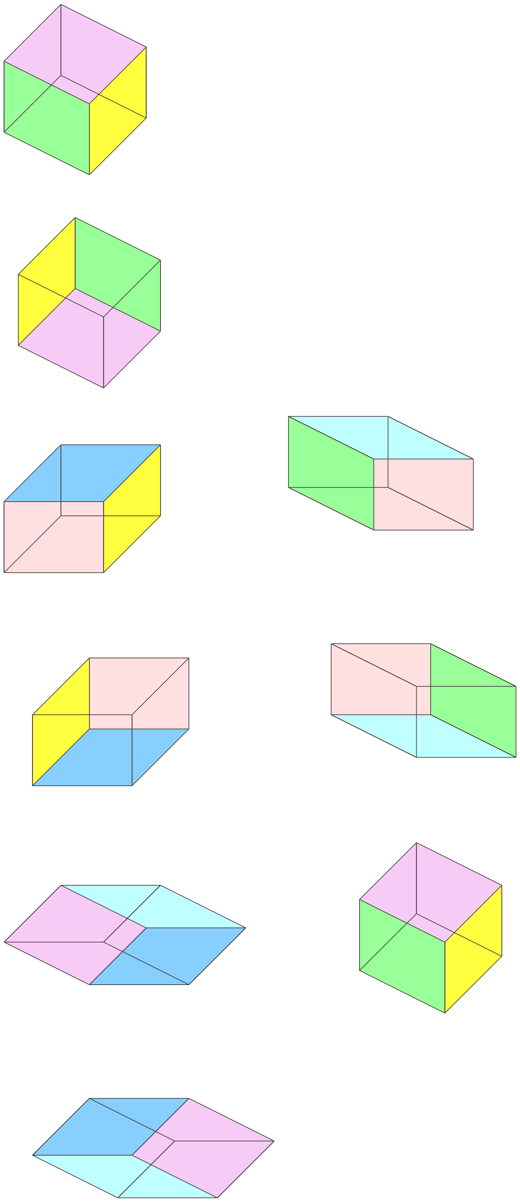


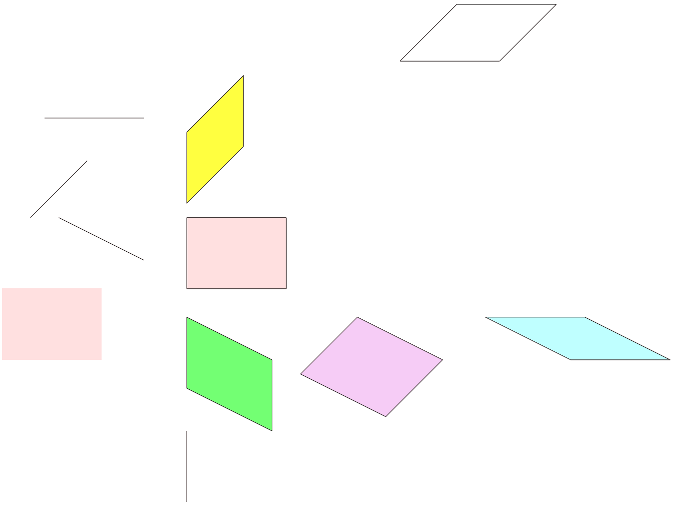


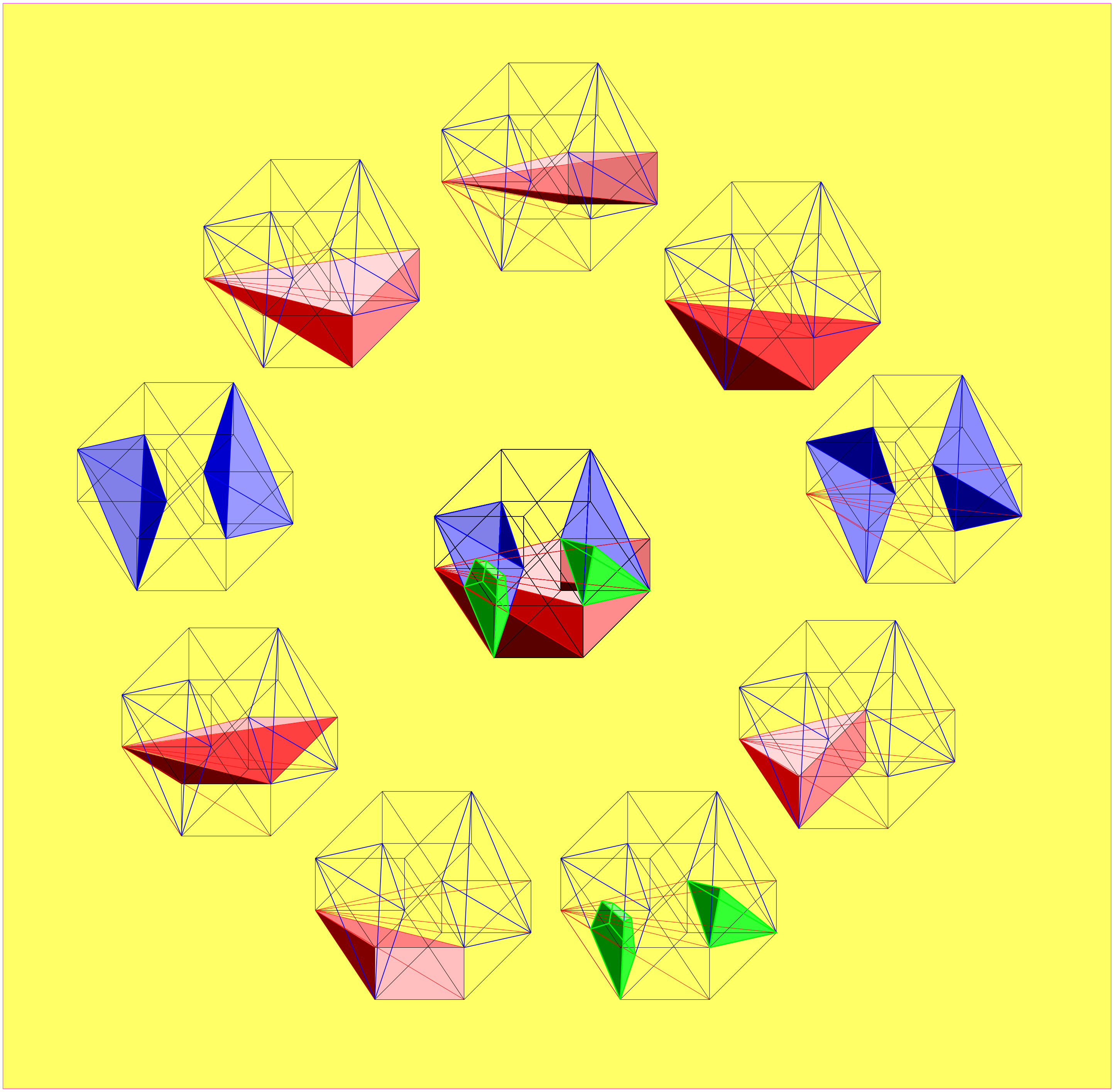


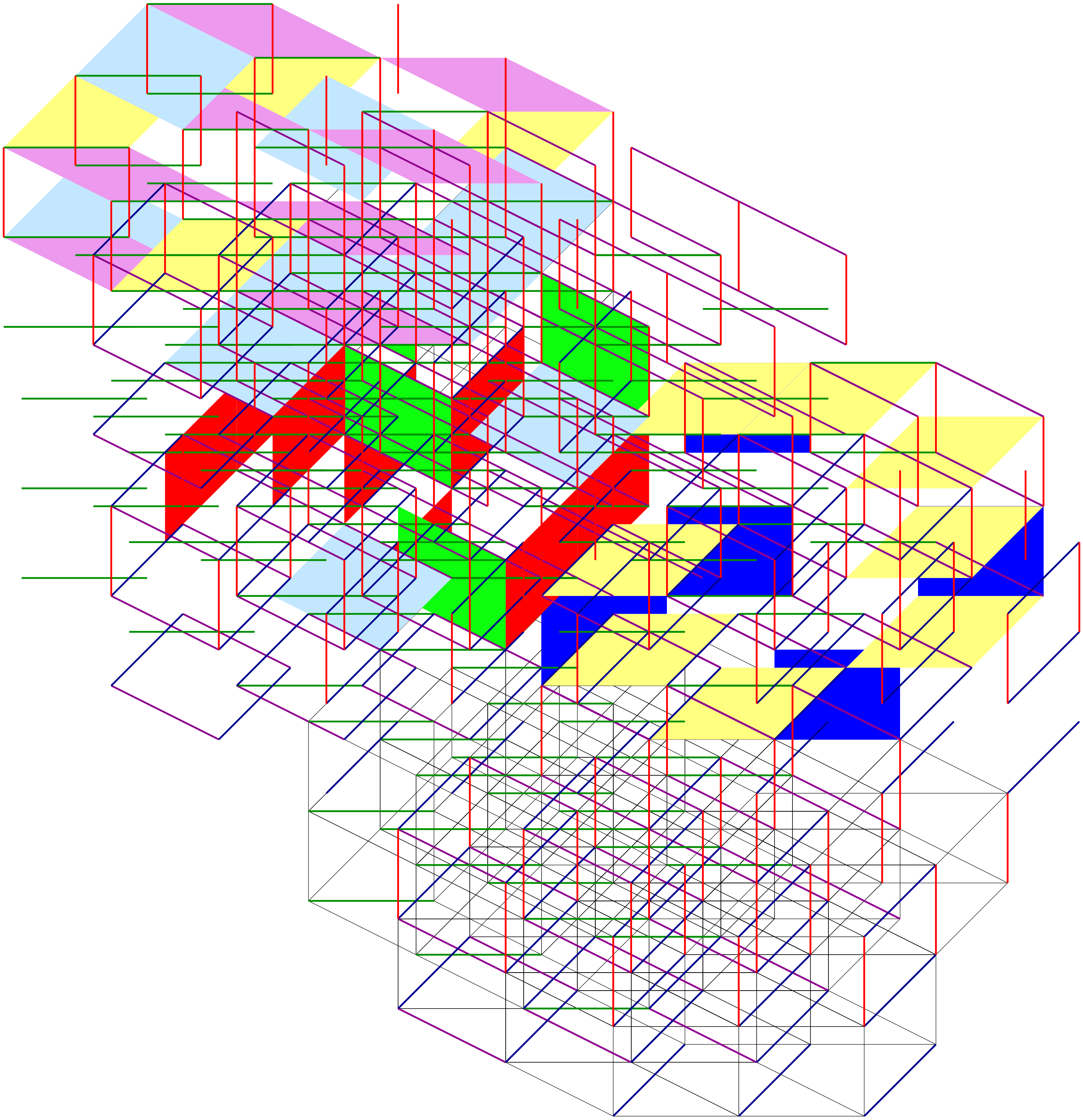


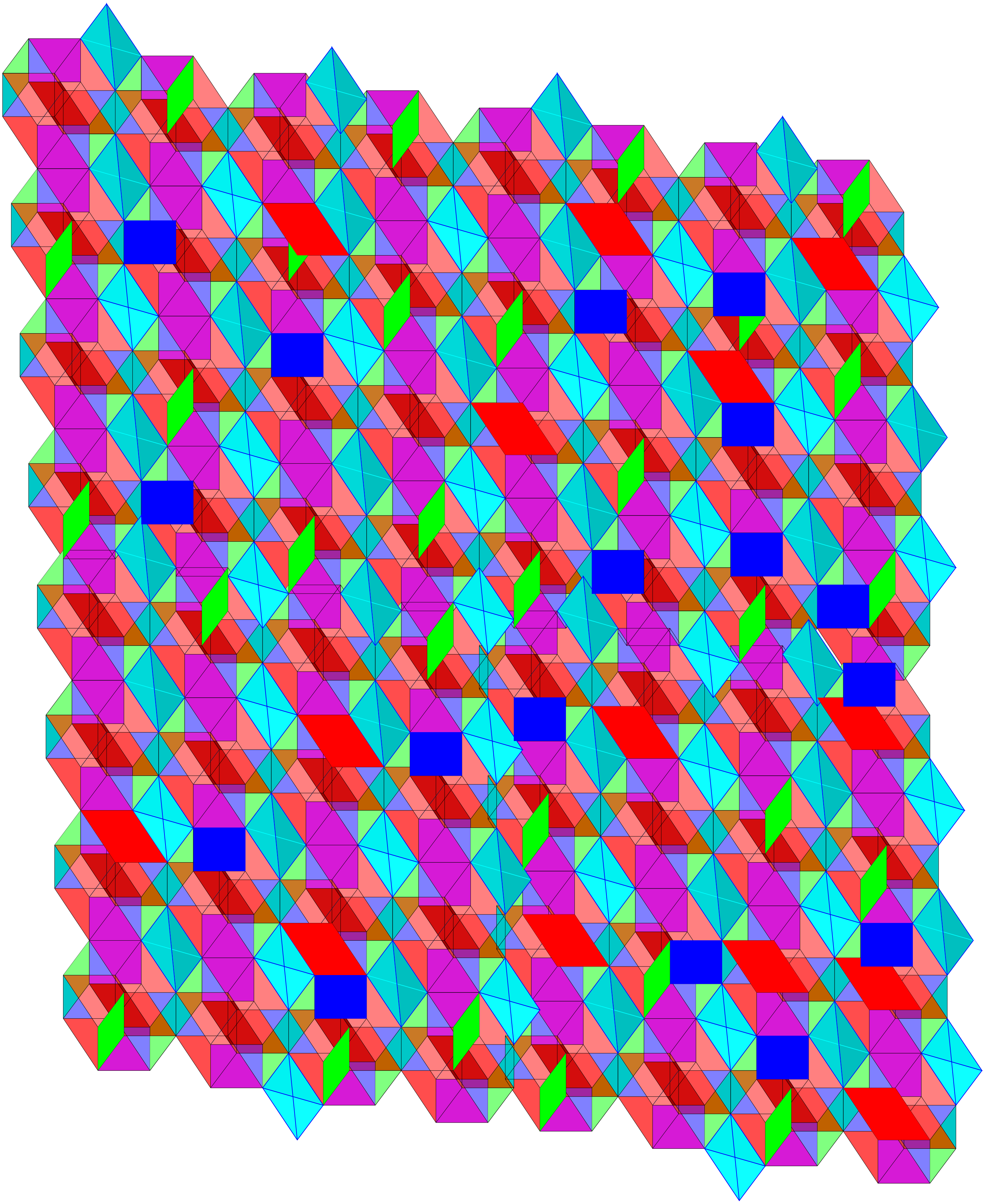


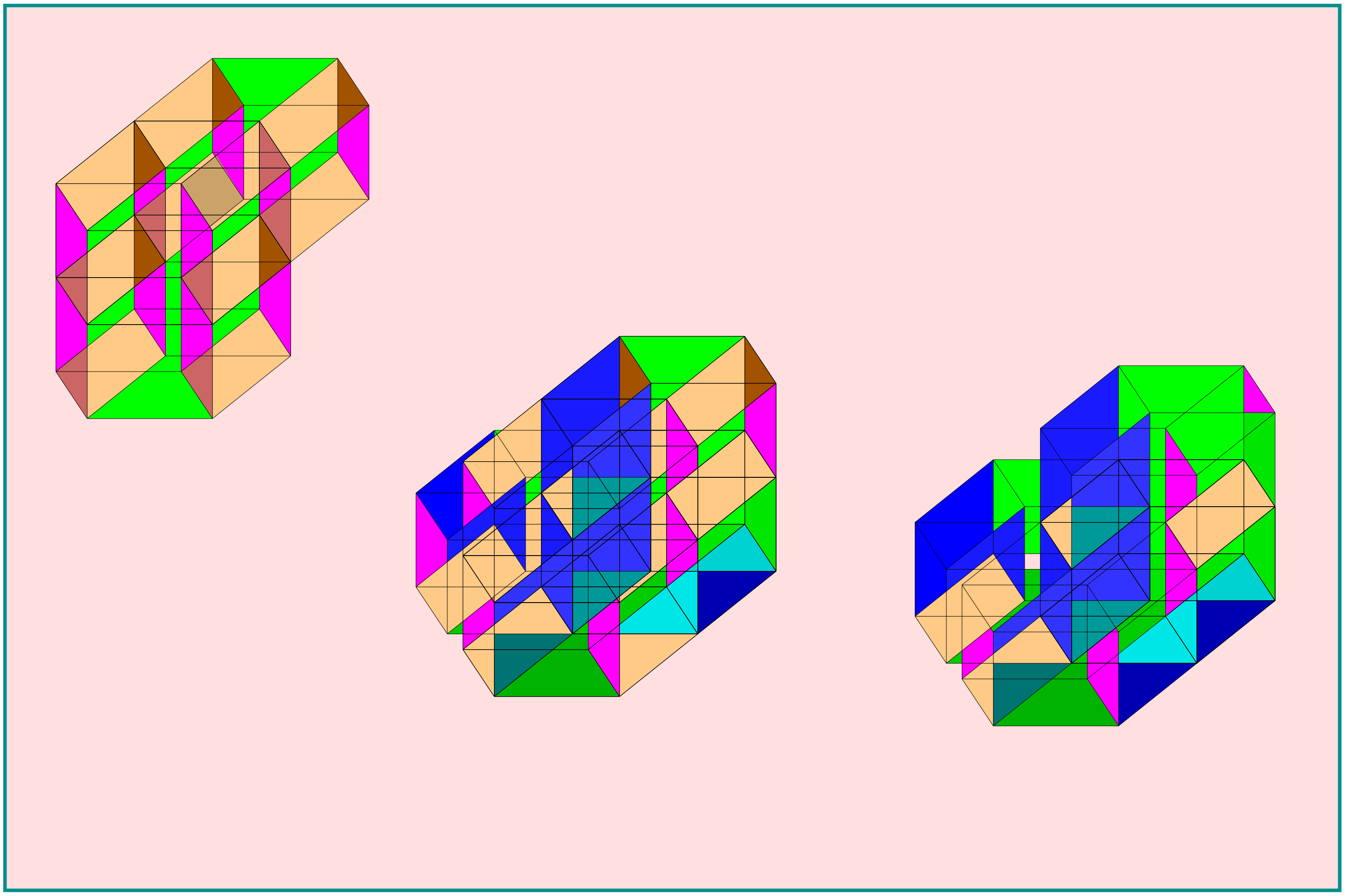


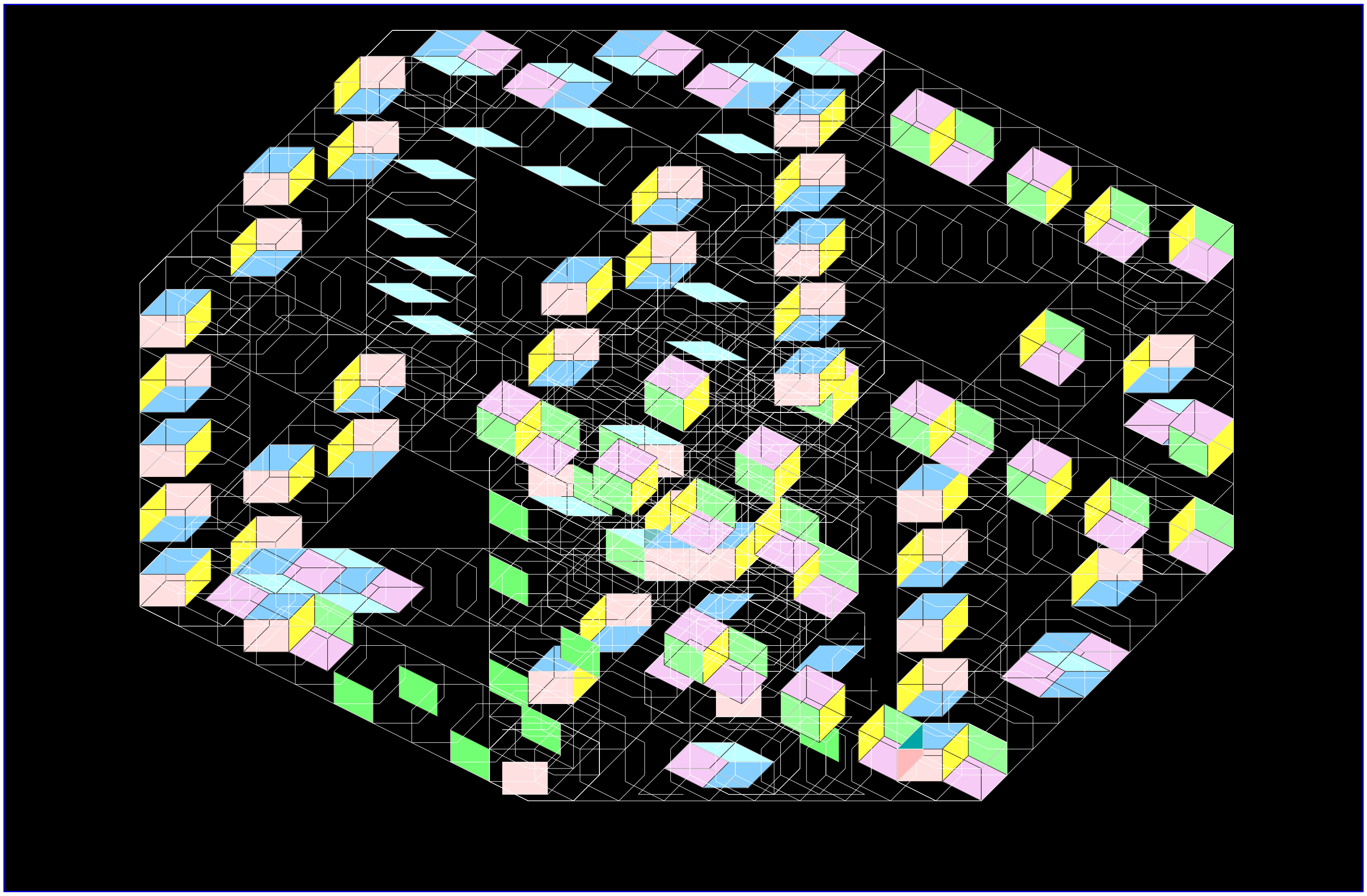


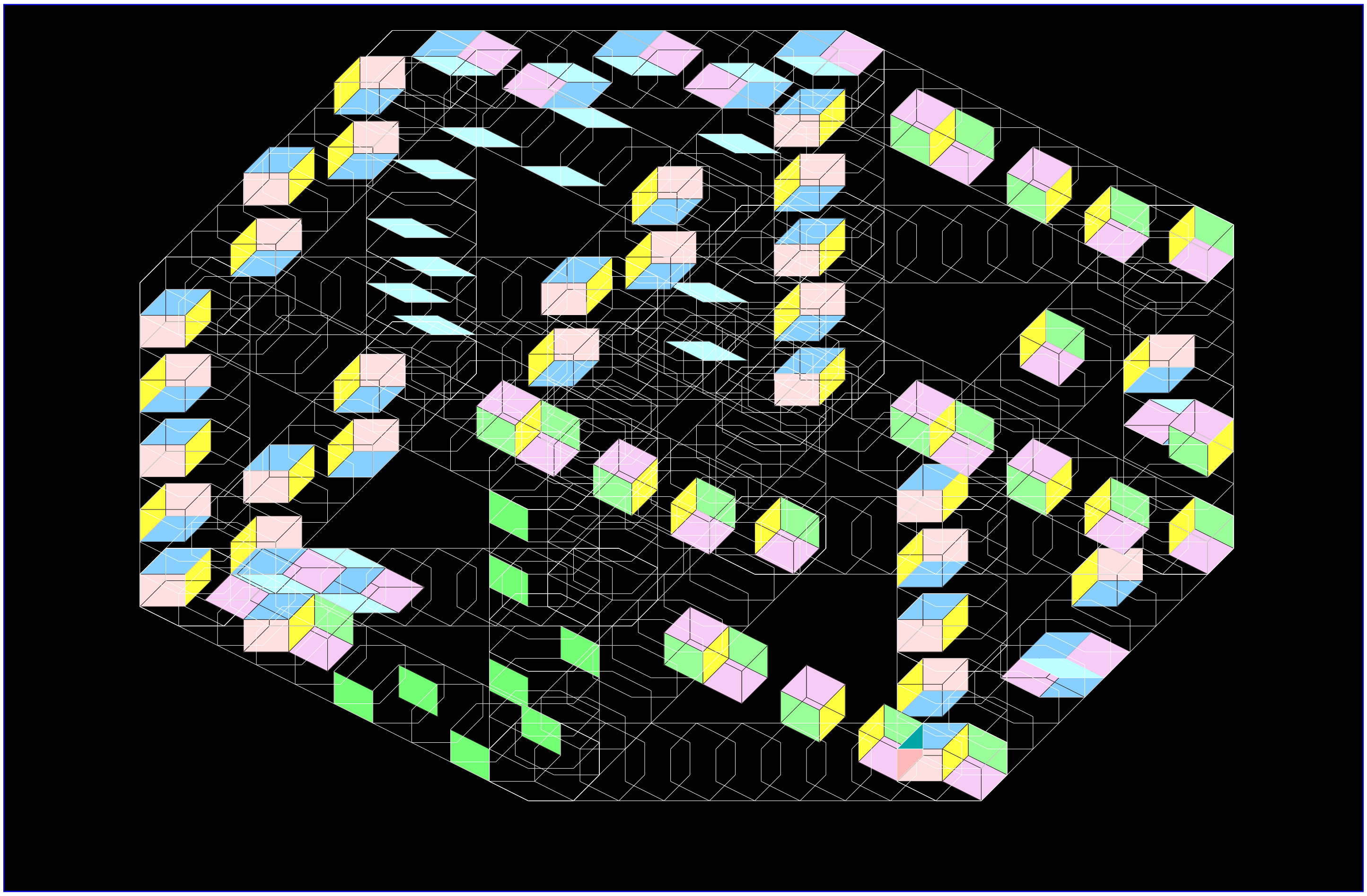


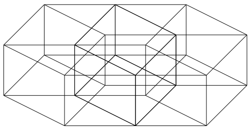
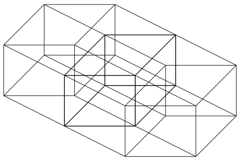
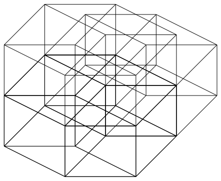
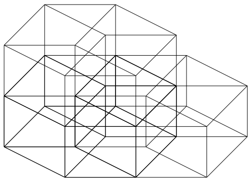
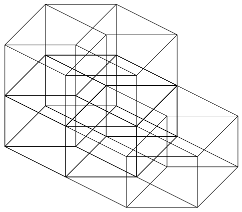
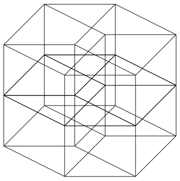
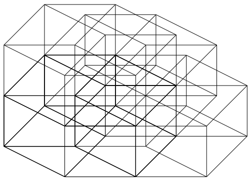
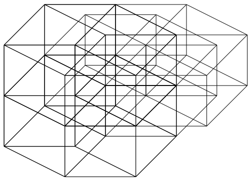
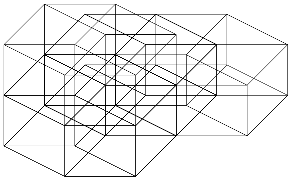










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