I want next to tell you a story about a chain letter.

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The next story begins from powers of 2; starting from $2^0 = 1$, $2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 16$, $2^5 = 32$, $2^6 = 64$, $2^7 = 128$, $2^8 = 256$, $2^9 = 512$, and $2^{10} = 1024$ are the first few values.

- chess board

- chess board

- 1 grain of rice for the first square

- chess board
- 1 grain of rice for the first square
- 2 grains of rice for the second square

- chess board
- 1 grain of rice for the first square
- 2 grains of rice for the second square
- 4 grains of rice for the third square

- chess board
- 1 grain of rice for the first square
- 2 grains of rice for the second square
- 4 grains of rice for the third square
- etc

- chess board
- 1 grain of rice for the first square.
- 2 grains of rice for the second square.
- 4 grains of rice for the third square.
- etc
- 2^{63} grains of rice on the 64th square.

This gives

$$1 + 2 + 4 + \cdots + 2^{63} = 2^{64} - 1$$

grains of rice. Estimate: $2^{10} = 1024$.

So $10^3 = 1000 < 2^{10}$.

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$$2^{64} = 2^4 \cdot (2^{10})^6 > 16 \times (10^3)^6 = 16 \times 10^{18}$$

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Or

$$2^{64} - 1 > 16,000,000,000,000,000,000.$$

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Rice costs $1/12,500 per grain.

$$1 + 2 + 4 + \cdots + 2^{63} = 2^{64} - 1$$

grains of rice.

Cost of Chess board $= \$3 \times 10^{15}$.

Chess is an ancient game, and I presume that there should be interest charged on the debt to the inventor. This too, is an exponential growth problem, but I digress.

Other things that powers of 2 count.

Other things that powers of 2 count. Certainly the whole numbers from 0 to $2^n - 1$ can be counted by 2^n.

Binary Notation:

n	Binary representation
0	$(000)_2$
1	$(001)_2$
2	$(010)_2$
3	$(011)_2$
4	$(100)_2$
5	$(101)_2$
6	$(110)_2$
7	$(111)_2$

The digits in 6, for example. indicate that $6 = 1 \times 2^2 + 1 \times 2^1 + 0 \times 2^0$. And I have padded the initial portions with 0 so that we can see that these powers don't appear.

The binary representation of a number indicates which powers of 2 to include to express the number. So the number 6 requires a 4 and a 2. The number 100 requires one 64, one 32, and one 4. So $100 = (1100100)_2$.

Another thing that is counted by powers of 2 is the collection of subsets of a set of fixed size.

Another thing that is counted by powers of 2 is the collection of subsets of a set of fixed size.

For example, if we have a 2 element set, $\{1, 2\}$, then we have the following four subsets, $\{\}$ — the set with no elements, $\{1\}, \{2\}$ — the two sets with one element each, and the two element set $\{1, 2\}$. For a three element set, we have the following eight subsets, $\{\}$ $\{1\}, \{2\}, \{3\}$, $\{1, 2\}$,$\{1, 3\}$, $\{2, 3\}$, and $\{1, 2, 3\}$. Of course a set with one element has two subsets: $\{\}$ $\{1\}$. And the empty set has only one subset.

1

1

1 1

1

1 1

1 2 1

1

1 1

1 2 1

1 3 3 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

$$1 = 1$$

$$1 + 1 = 2$$

$$1 + 2 + 1 = 4$$

$$1 + 3 + 3 + 1 = 8$$

$$1 + 4 + 6 + 4 + 1 = 16$$

$$1 + 5 + 10 + 10 + 5 + 1 = 32$$

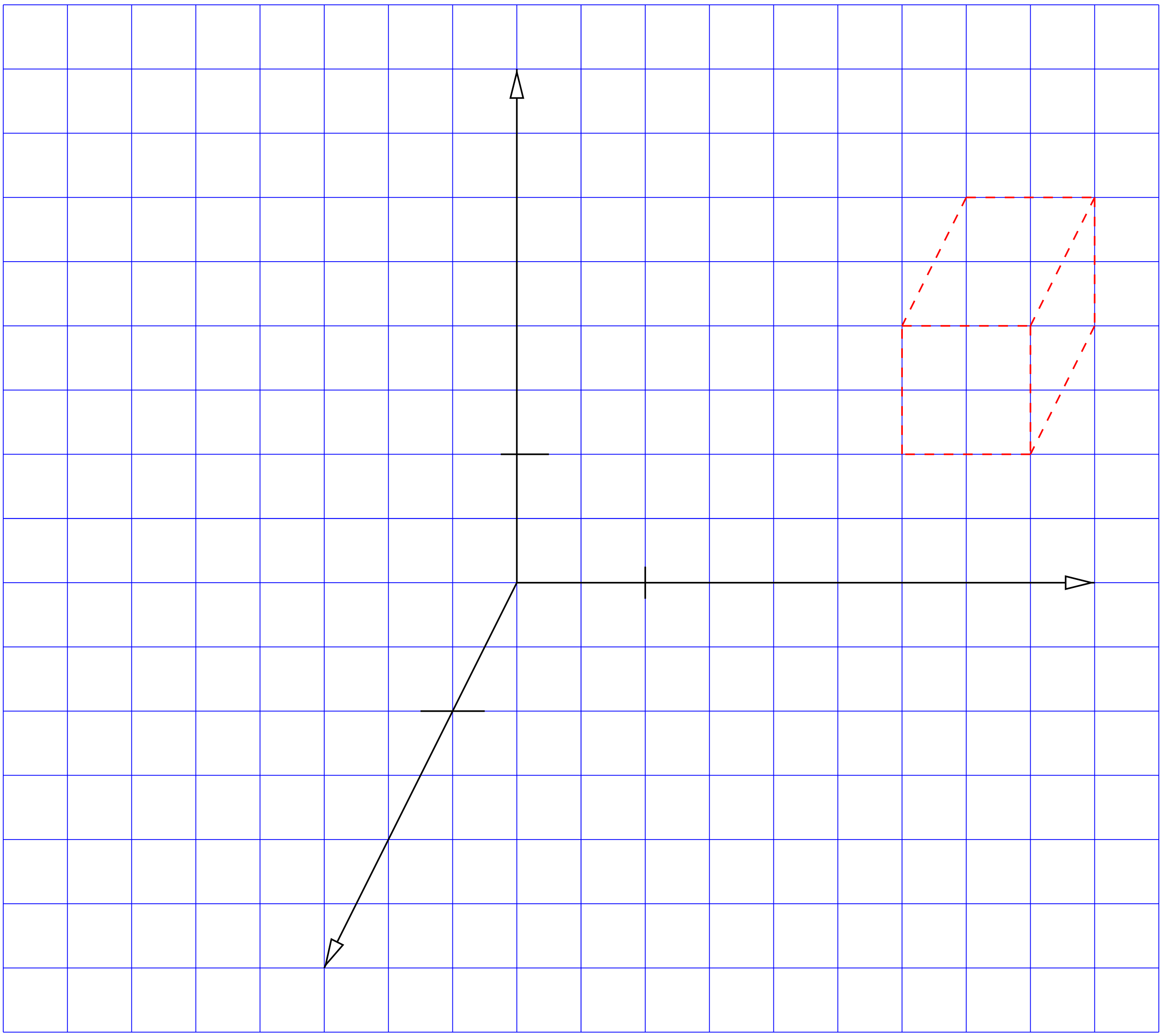
$$1 + 6 + 15 + 20 + 15 + 6 + 164$$

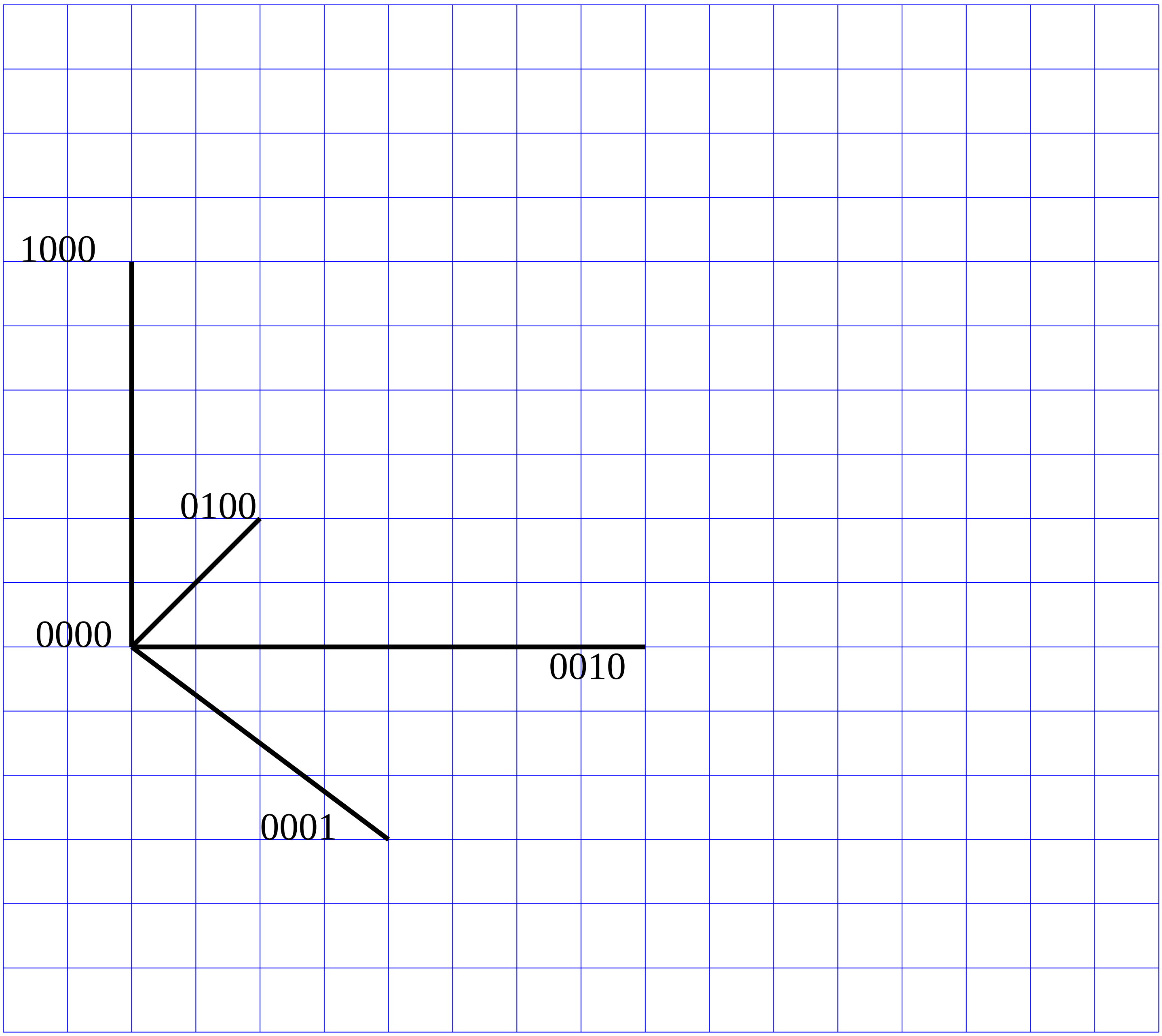
$$1 + 7 + 21 + 35 + 35 + 21 + 7 + 1 = 128$$

Pascal's triangle determines the coeficients of of $x^i y^j$ in the expansion of $(x+y)^n$.

Now why does two to a power count the number of subsets of a given set? Why can these subsets be arranged in size according to Pascal's triangle? Is there a convenient way of arranging these subsets — to keep track of them and relationships among them?

To answer these questions, I will begin in the most simple cases, and I will gradually build up the dimension ladder.

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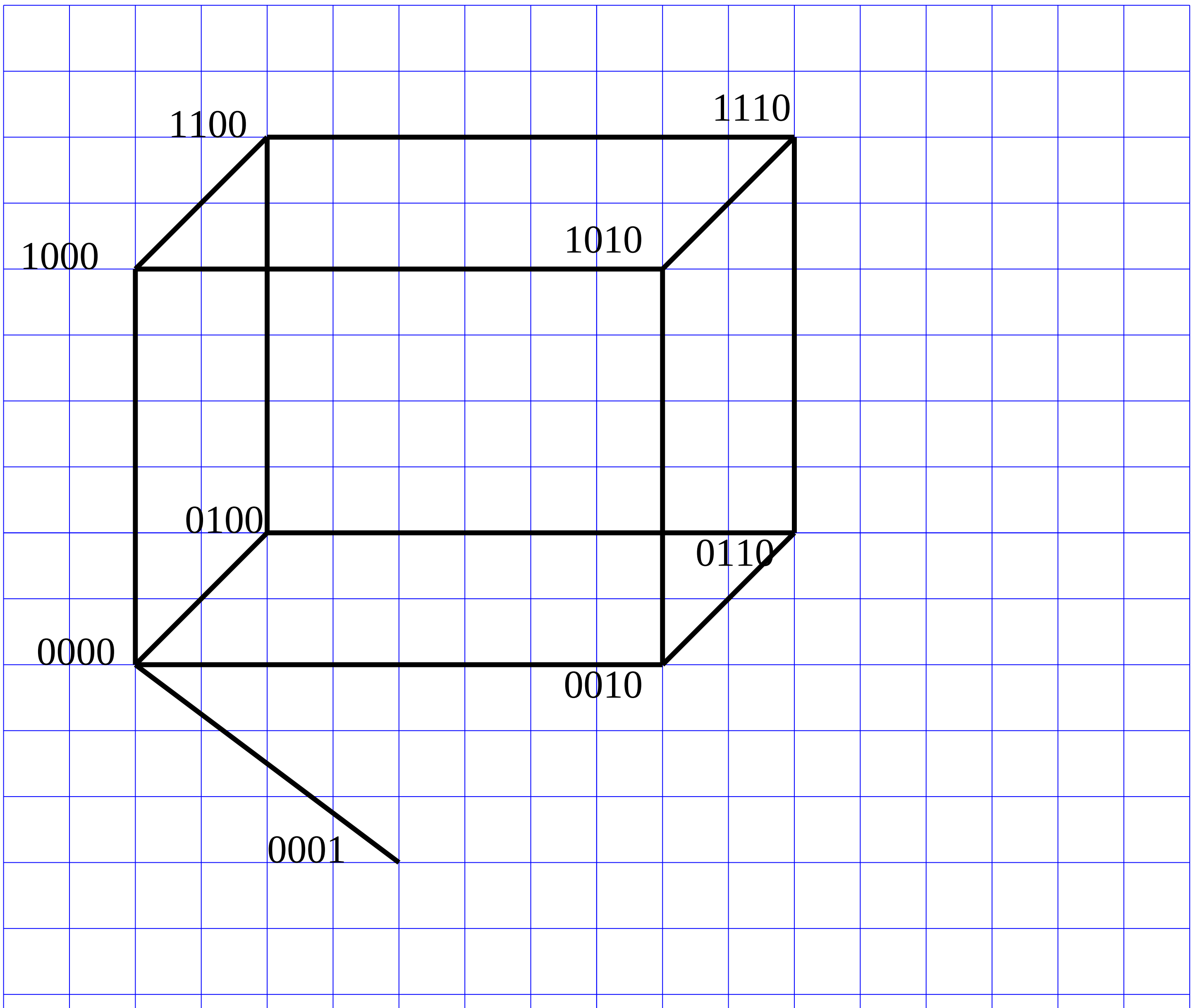
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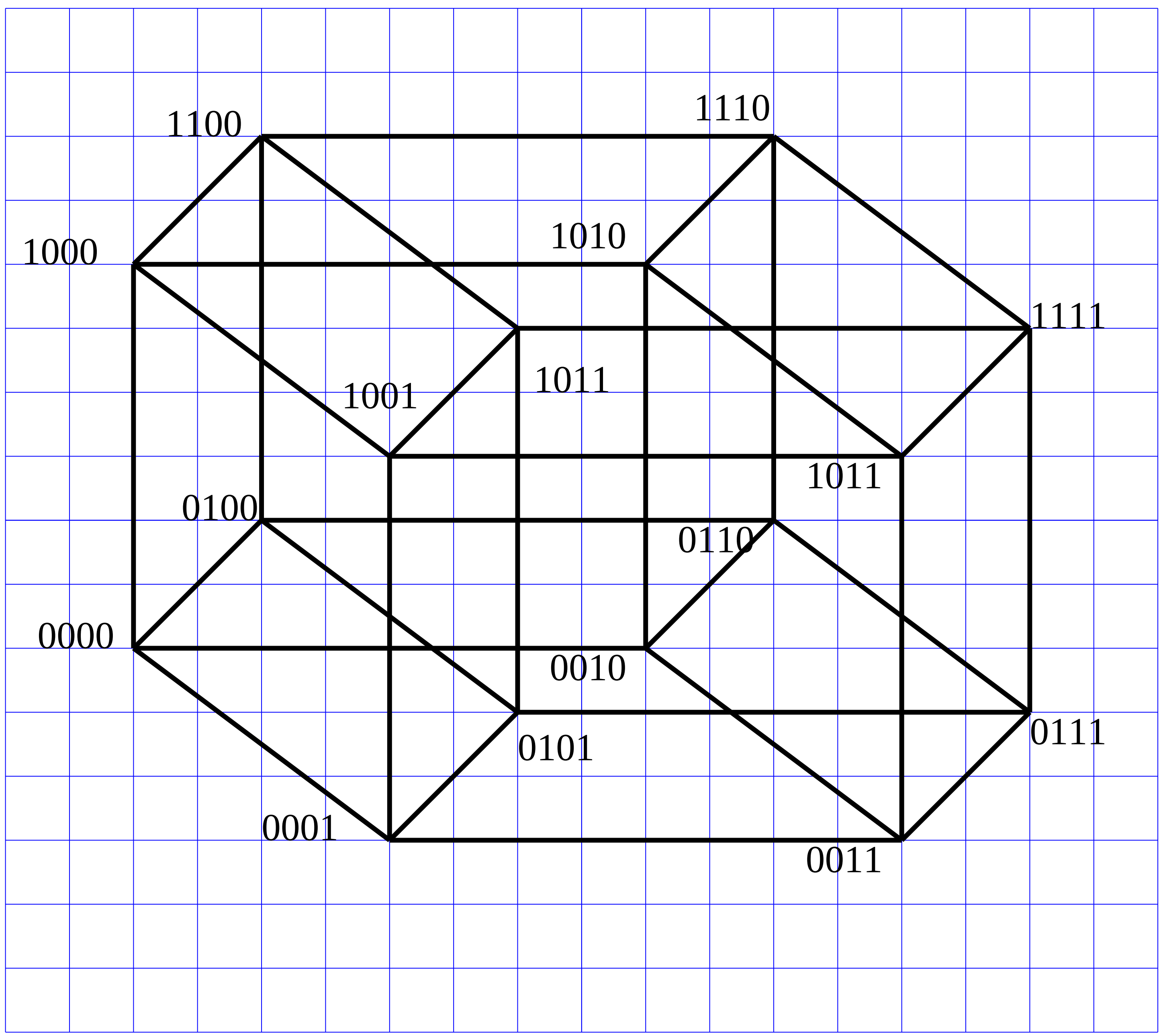

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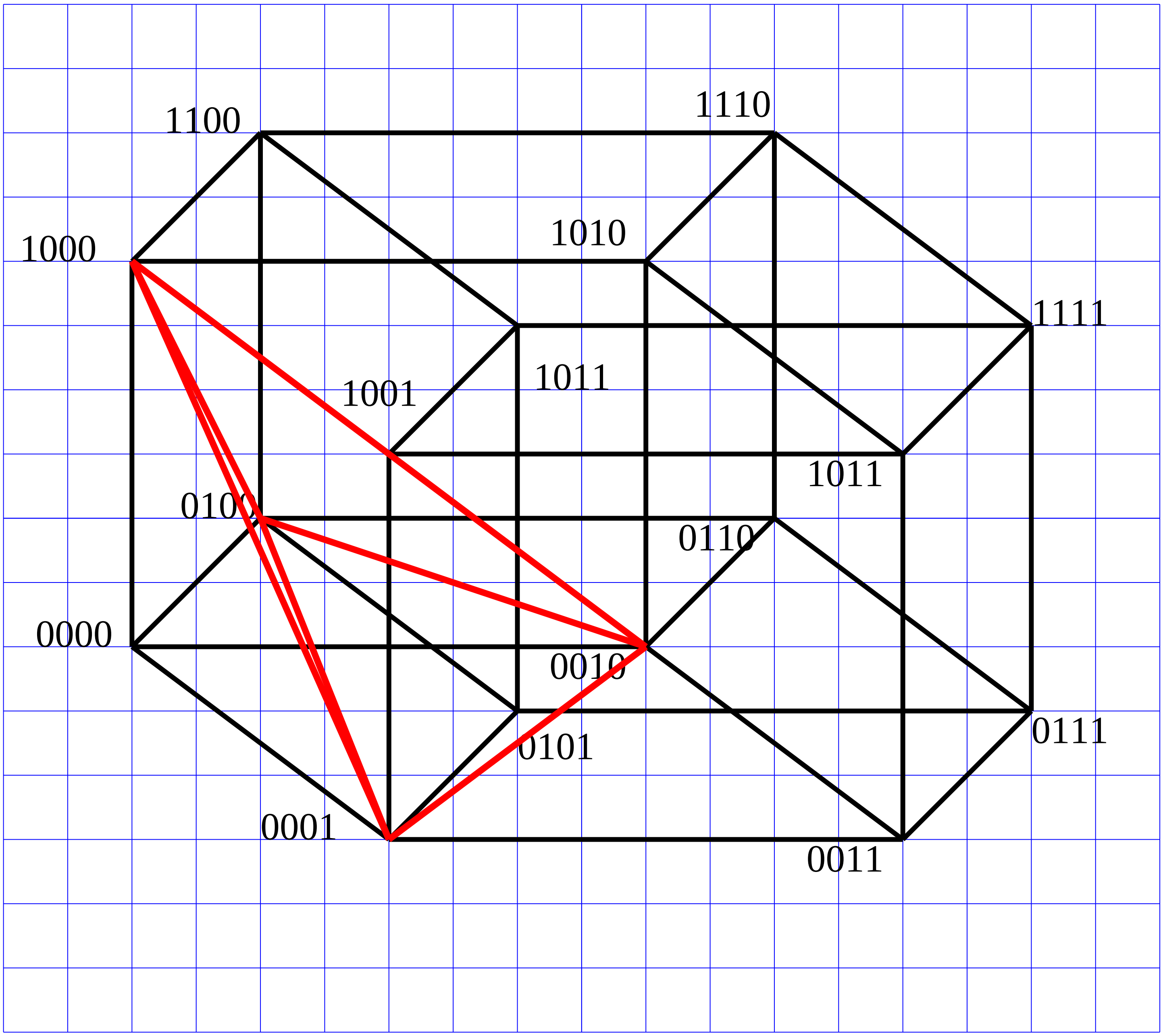
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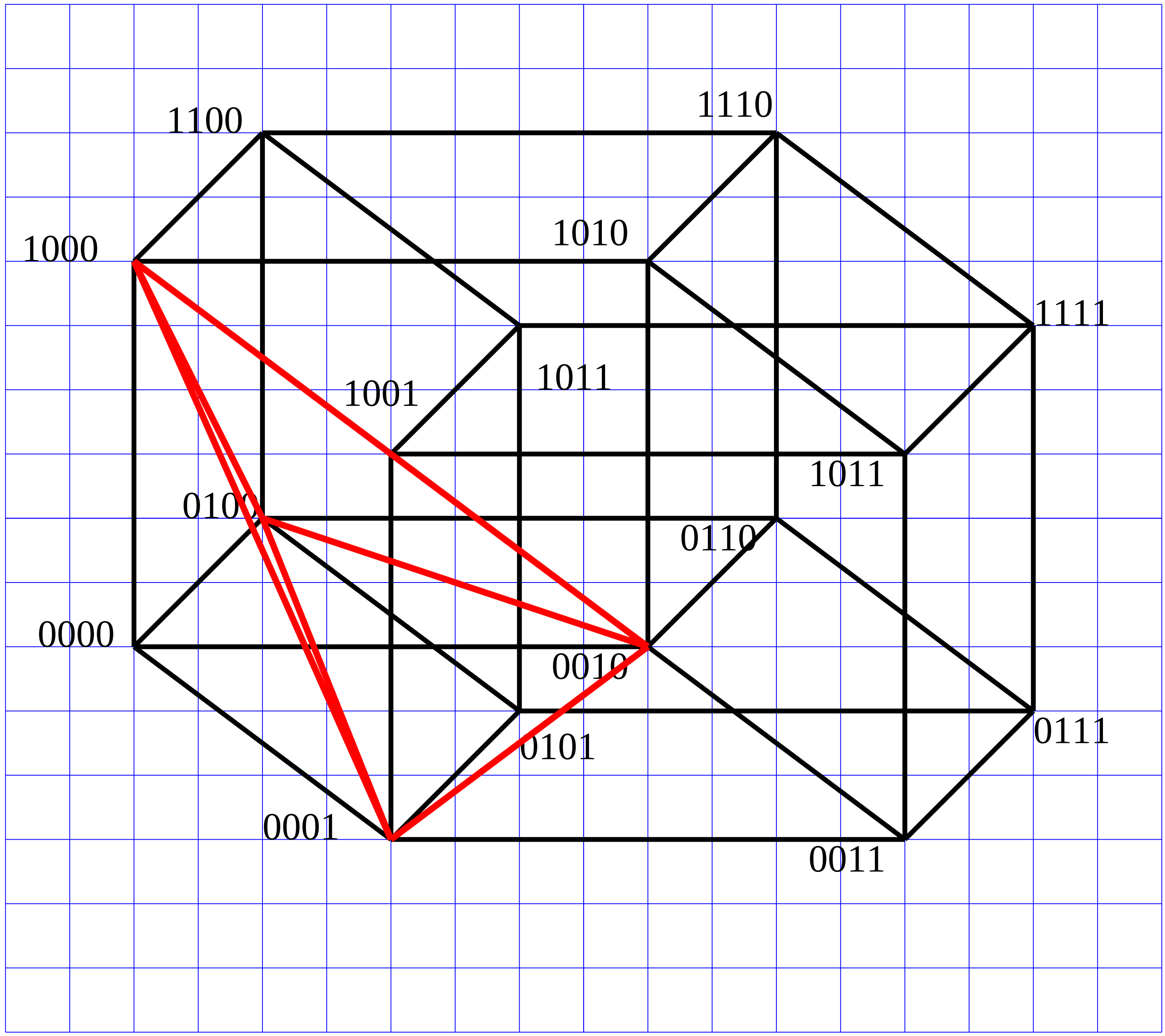
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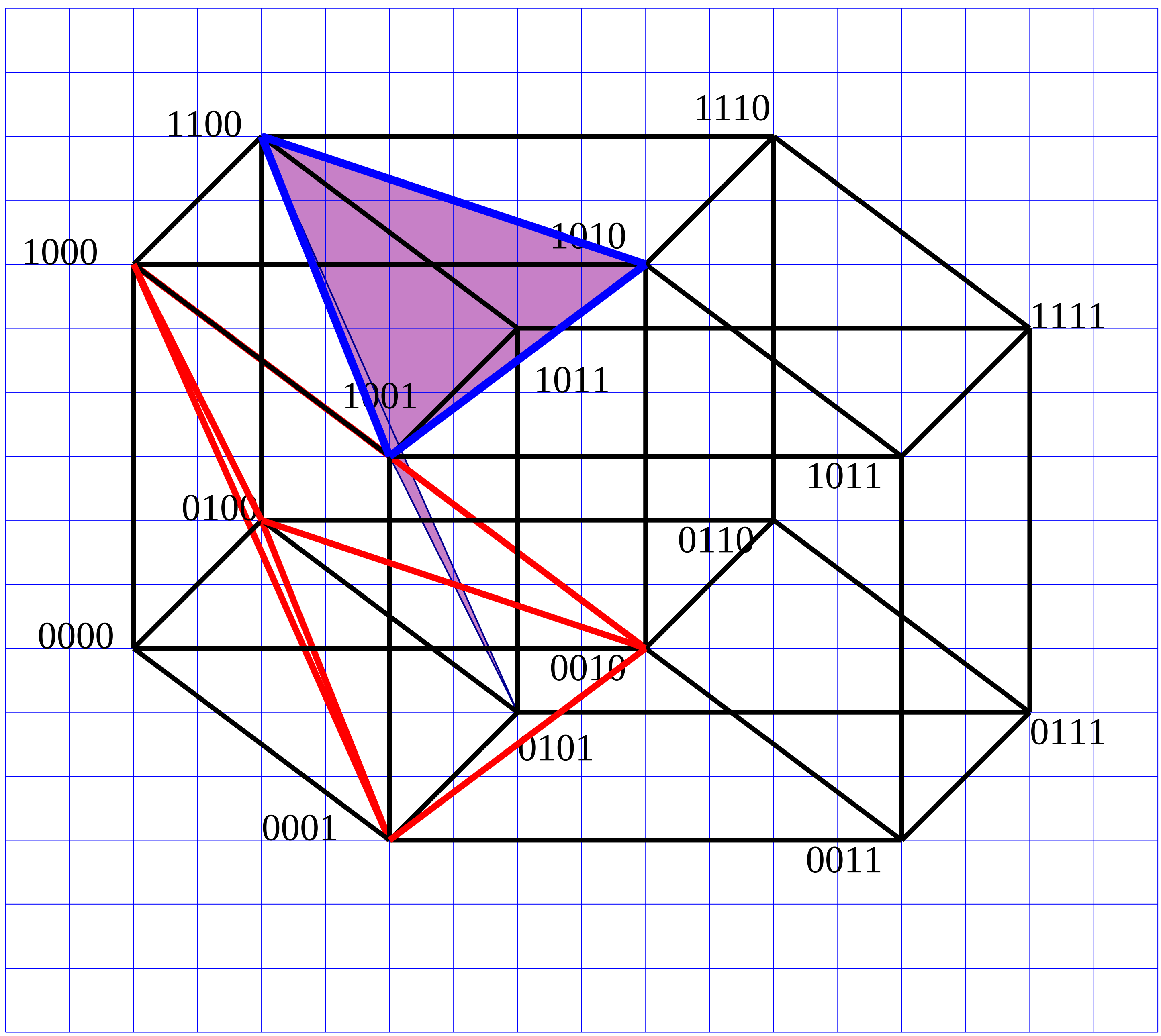
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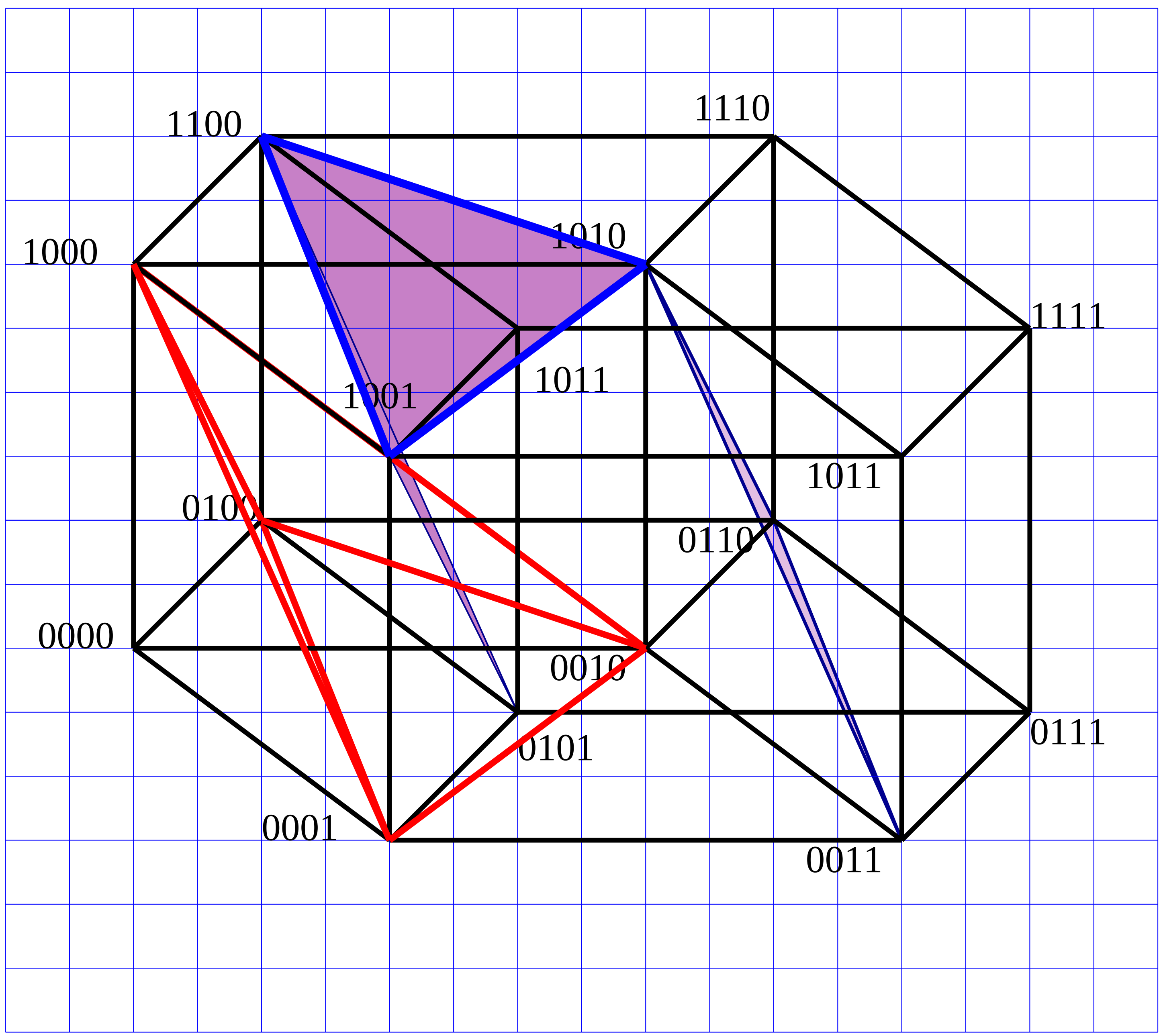
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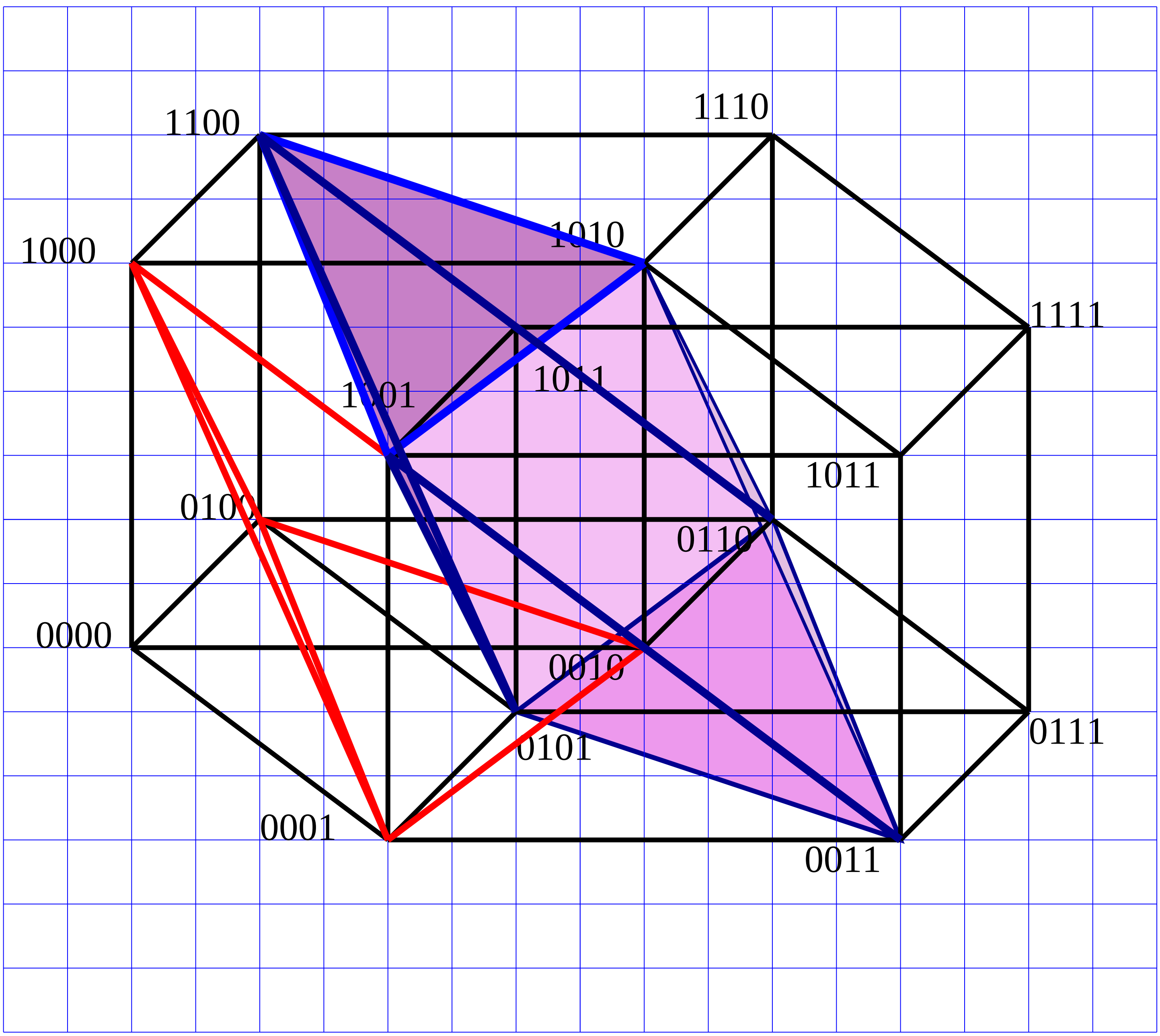
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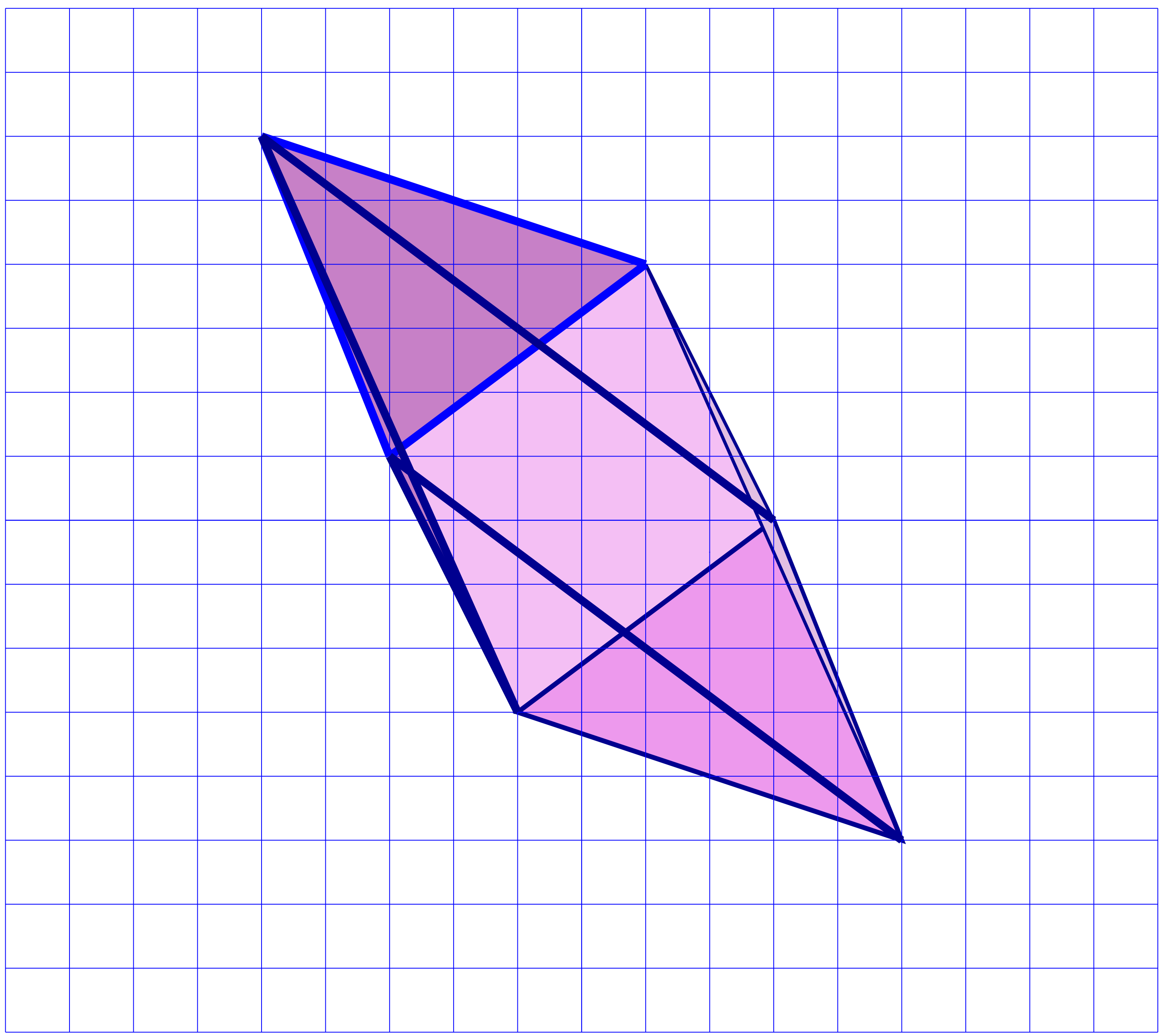

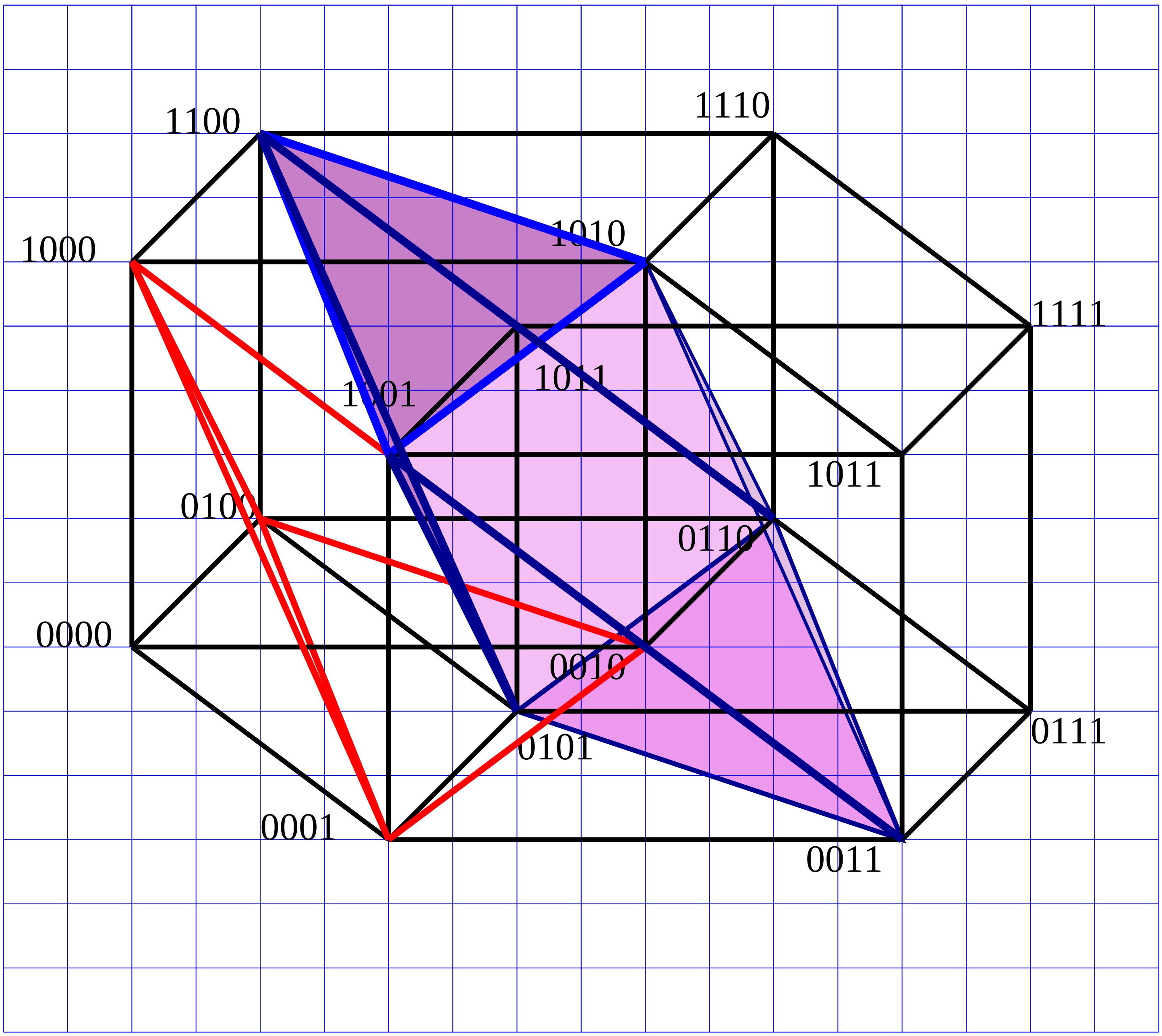
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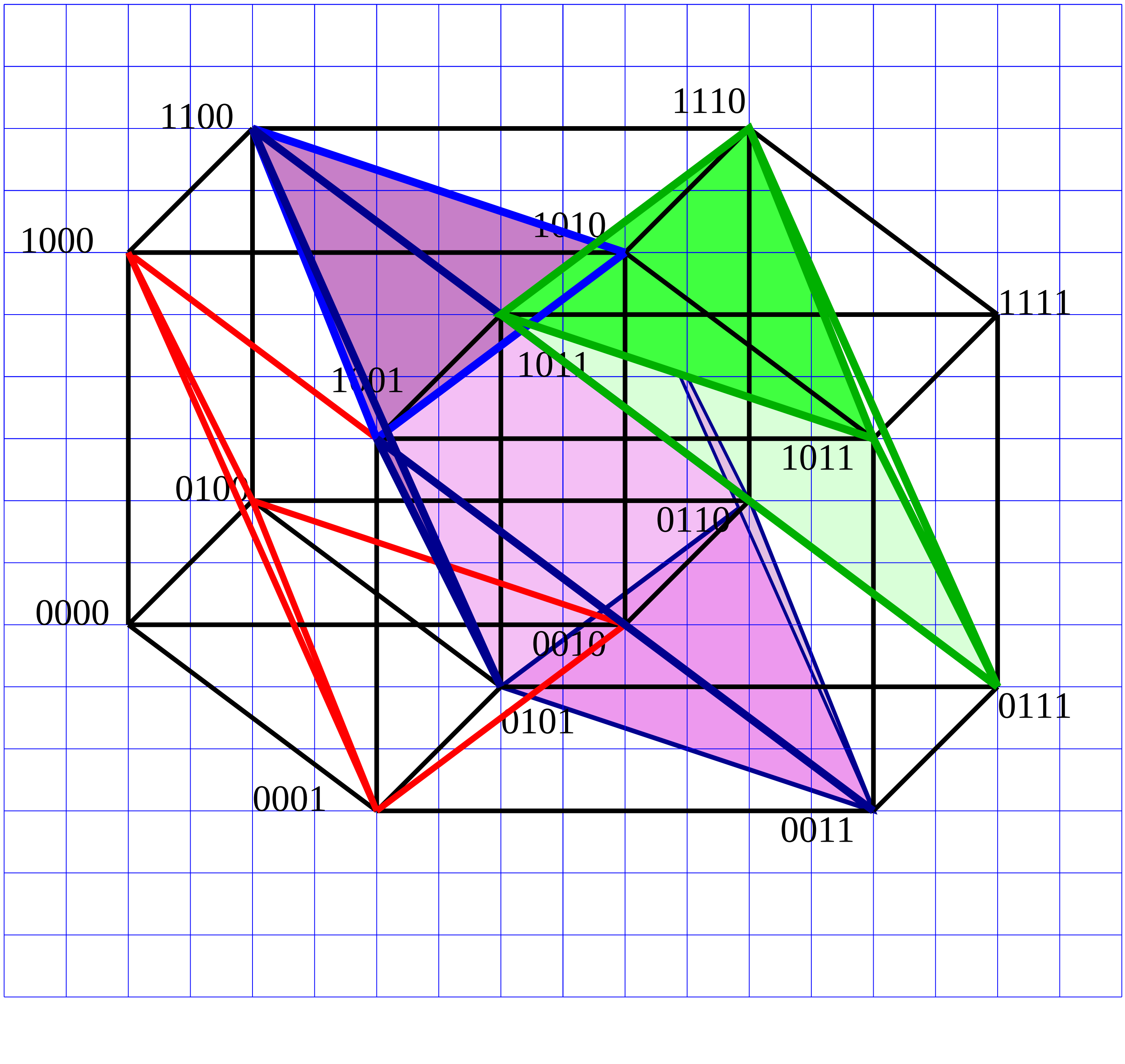
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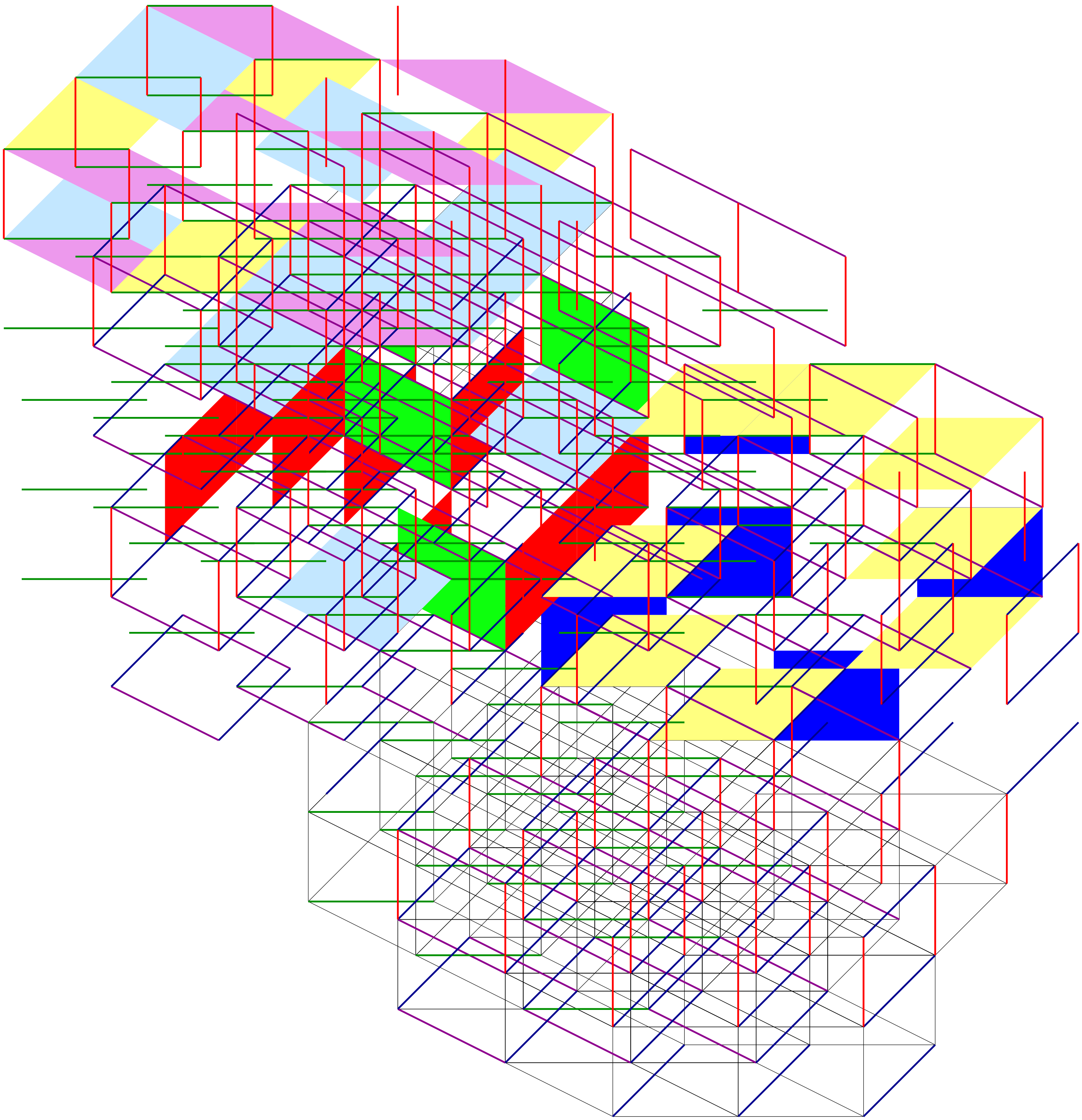

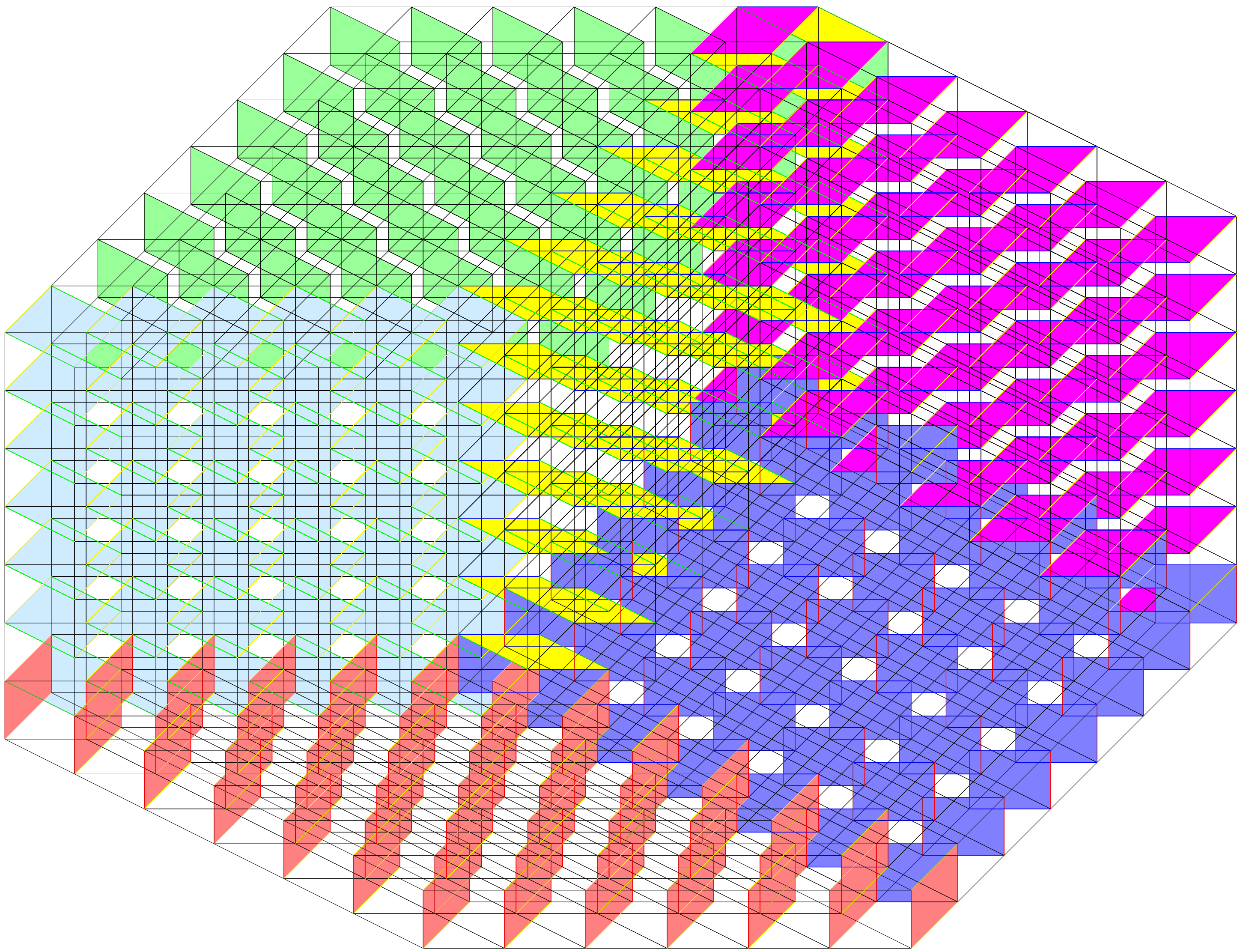

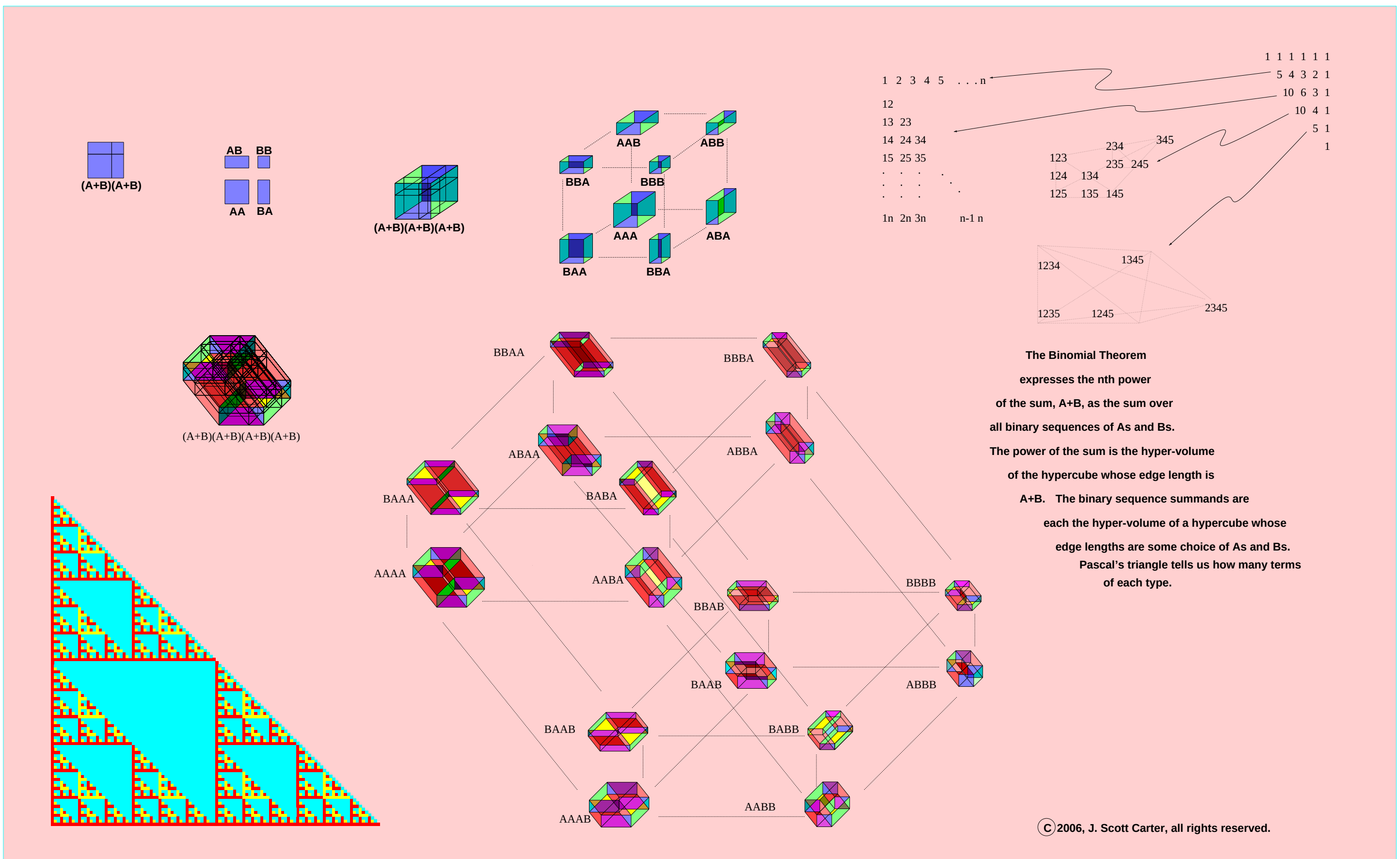
(A+B)(A+B)
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AA BA
(A+B)(A+B)(A+B)
AAB
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BBA
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AAA
ABA
BAA
BBA
1 2 3 4 5 . . . n
12
13 23
14 24 34
15 25 35
1n 2n 3n n-1 n
1 1 1 1 1 1
5 4 3 2 1
10 6 3 1
10 4 1
5 1
1
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124 134
125 135 145
234
235 245
345
1234
1345
1235
1245
2345
(A+B)(A+B)(A+B)(A+B)
BBAA
BBBA
ABAA
ABBA
BAAA
BABA
AAAA
AABA
BBAB
BBBB
BAAB
ABBB
BAAB
BABB
AAAB
AABB
The Binomial Theorem
expresses the nth power
of the sum, A+B, as the sum over
all binary sequences of As and Bs.
The power of the sum is the hyper-volume
of the hypercube whose edge length is
A+B. The binary sequence summands are
each the hyper-volume of a hypercube whose
edge lengths are some choice of As and Bs.
Pascal's triangle tells us how many terms
of each type.
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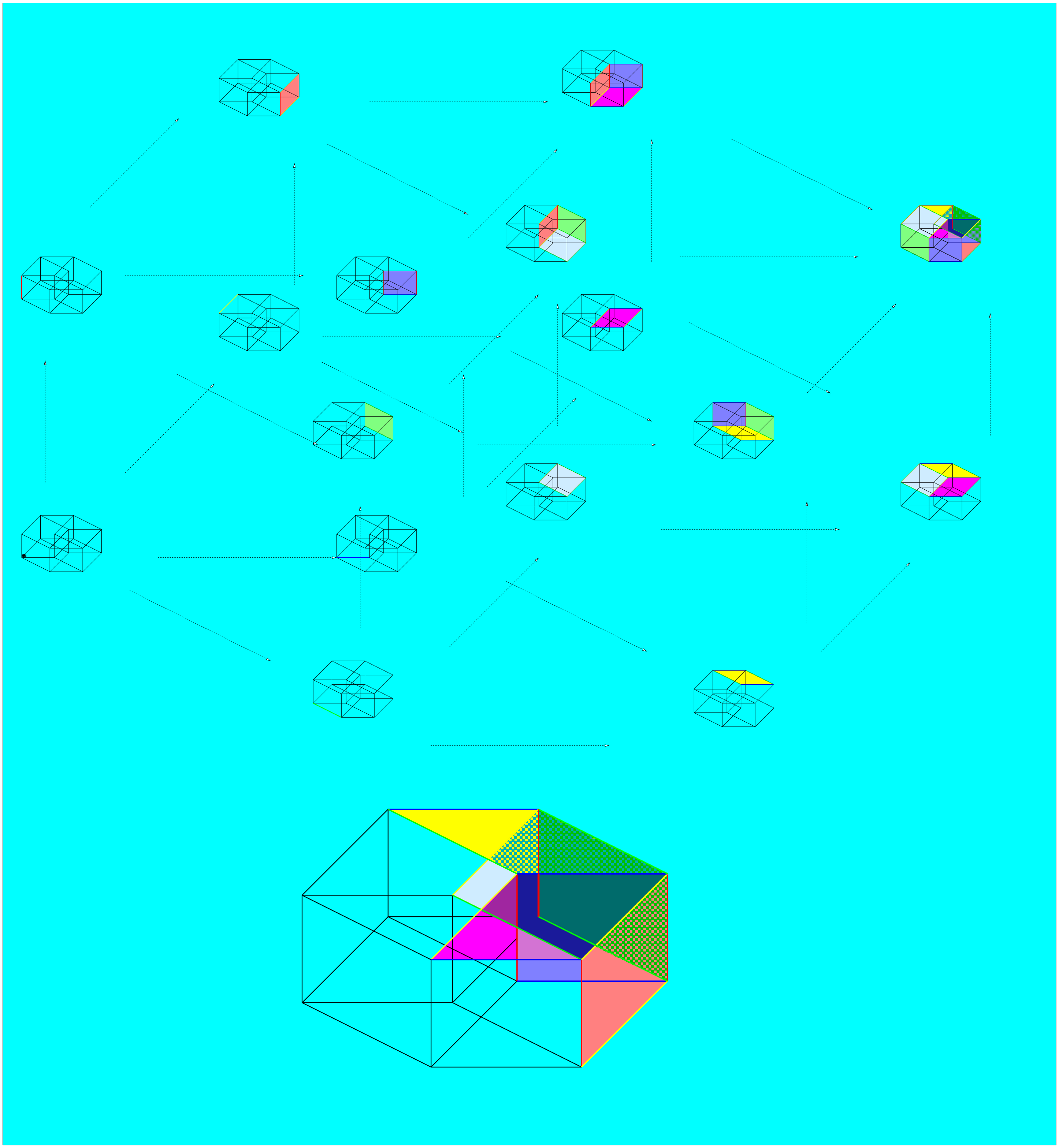
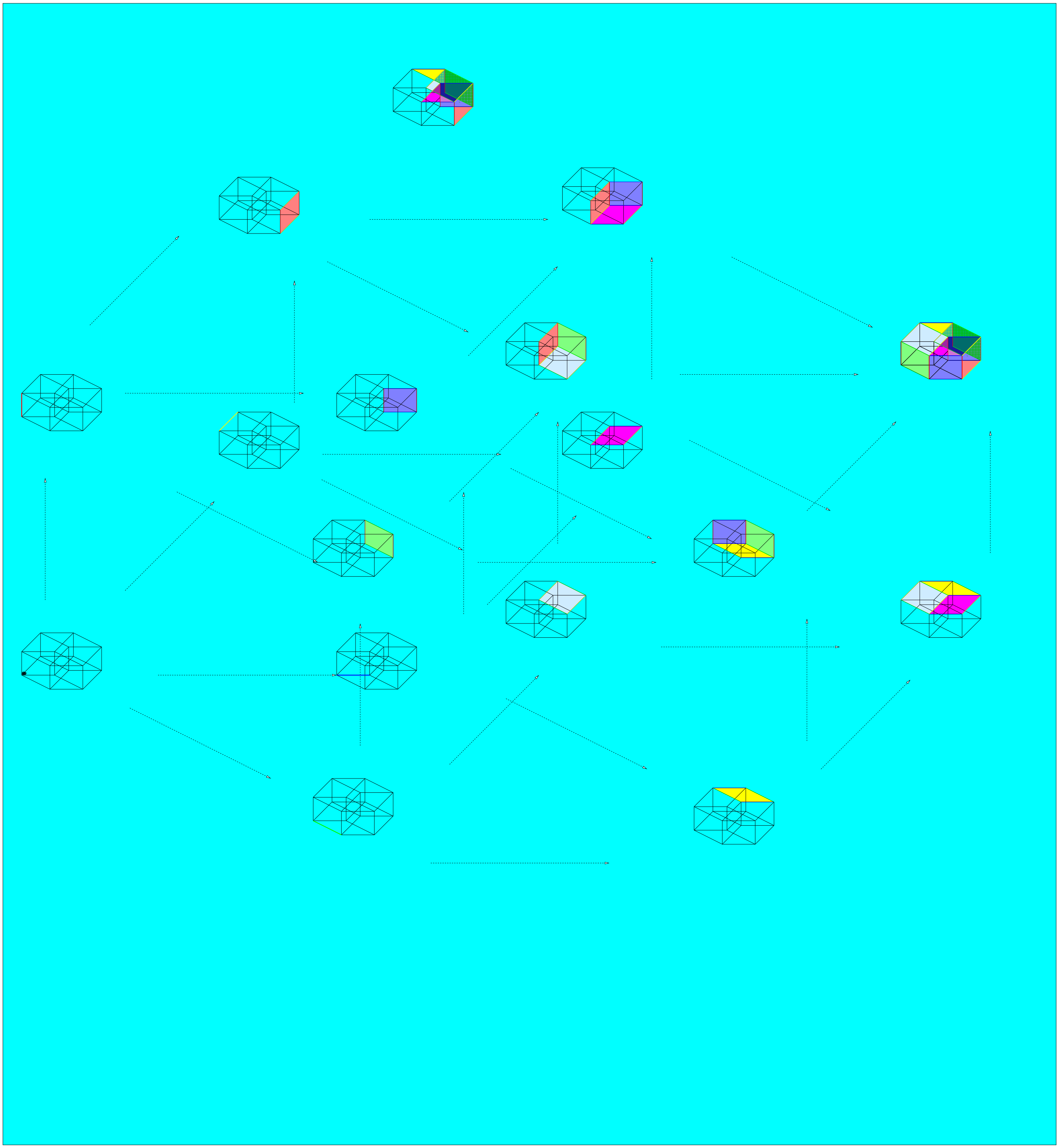
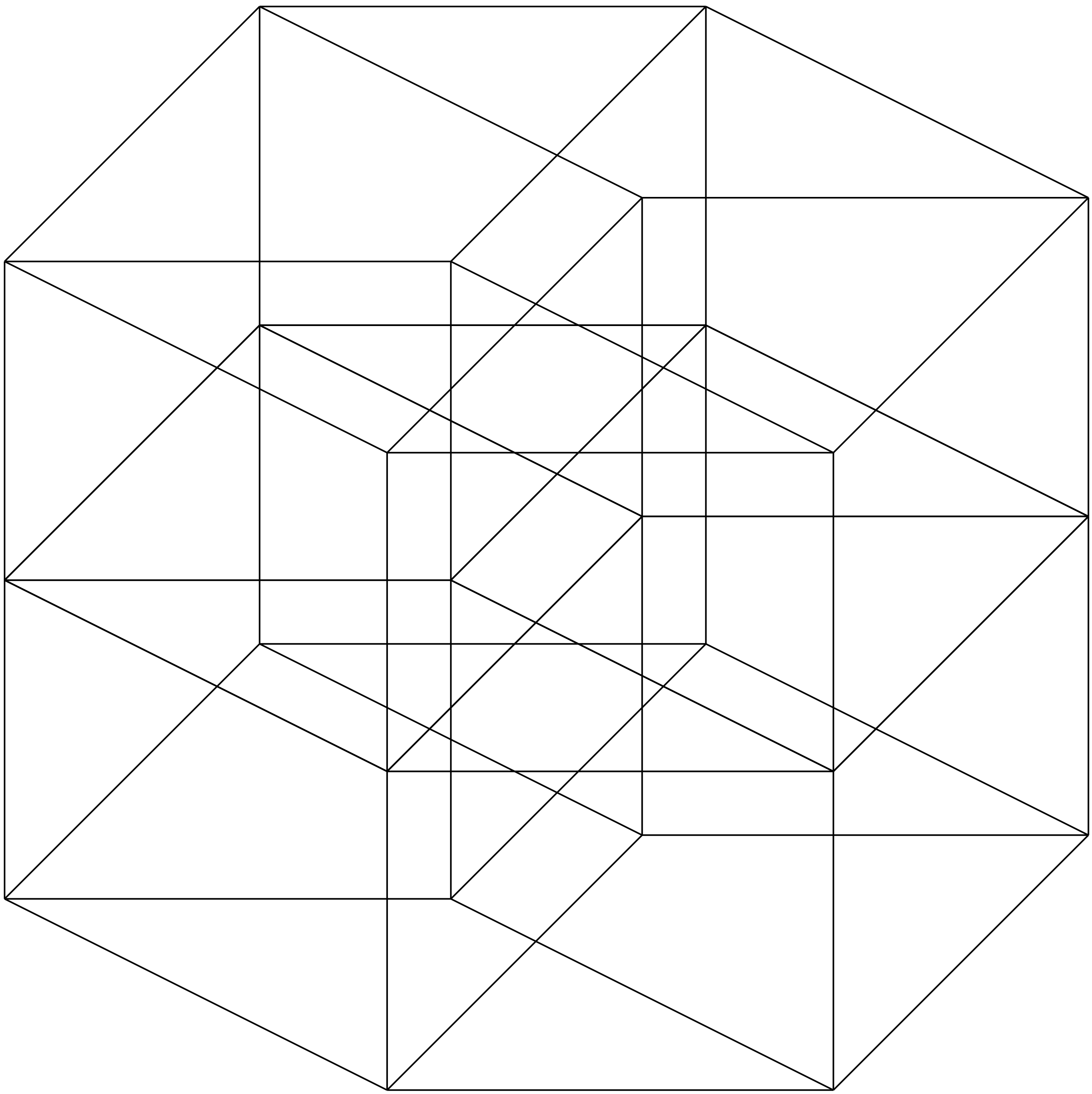
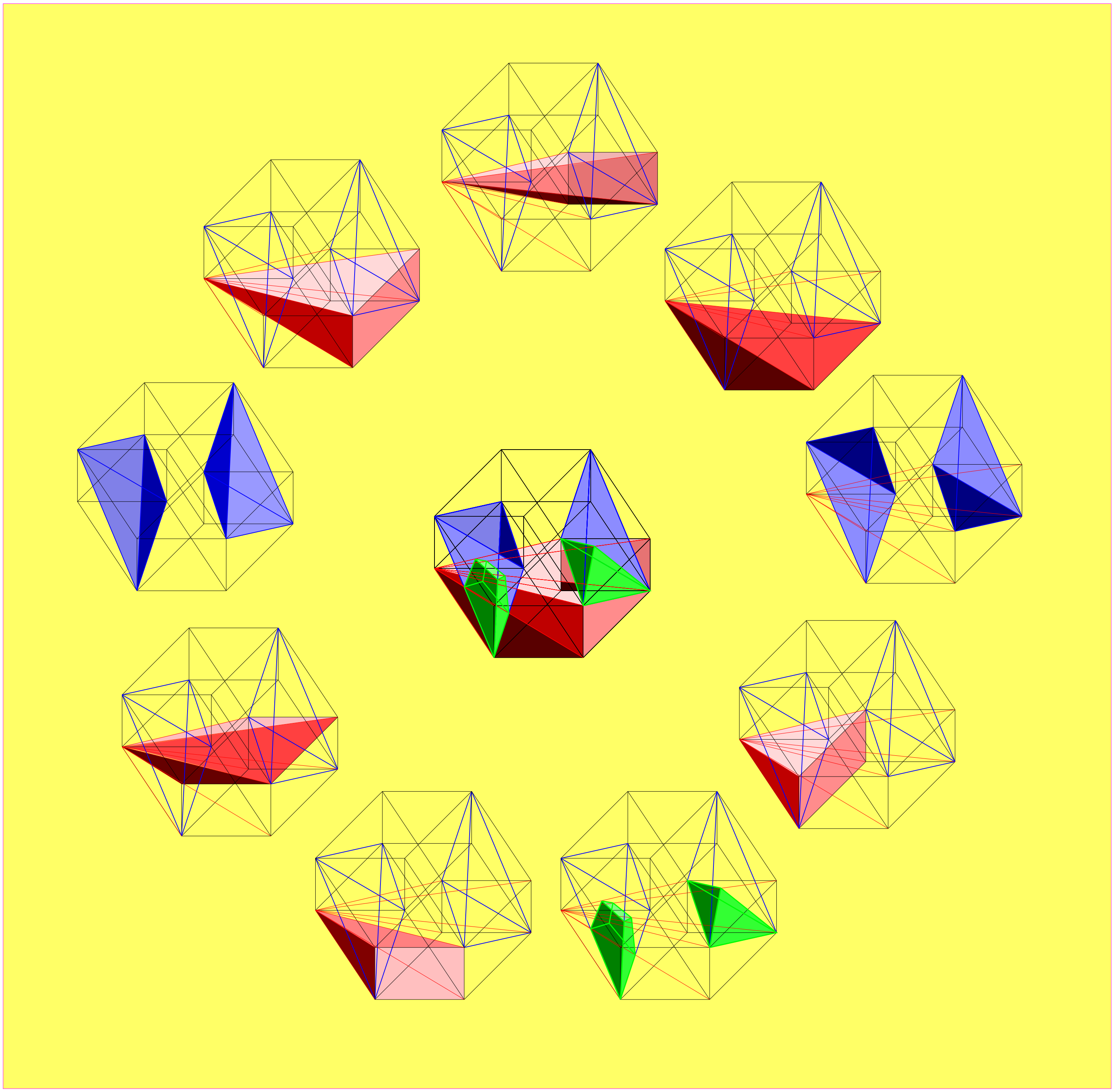
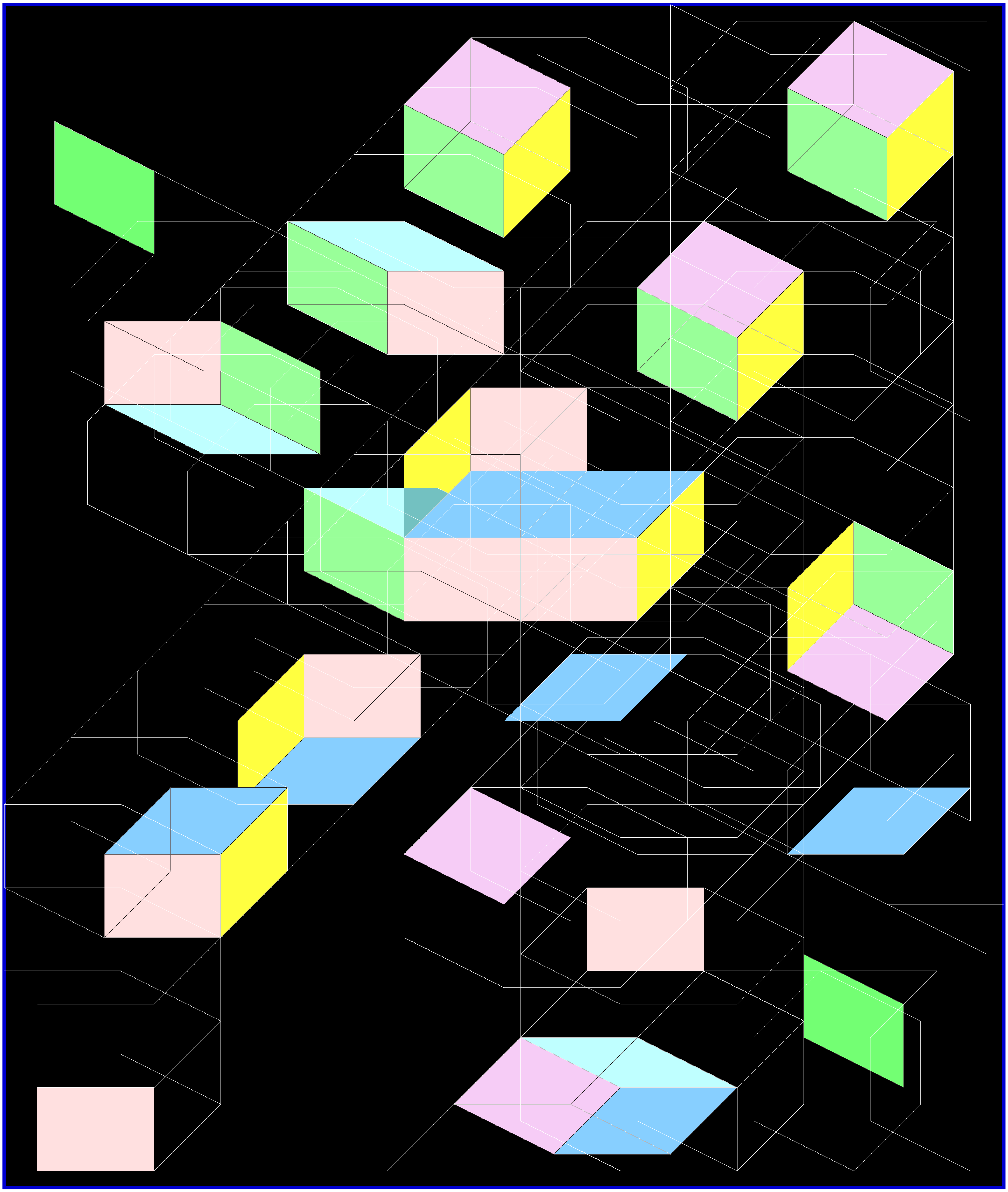
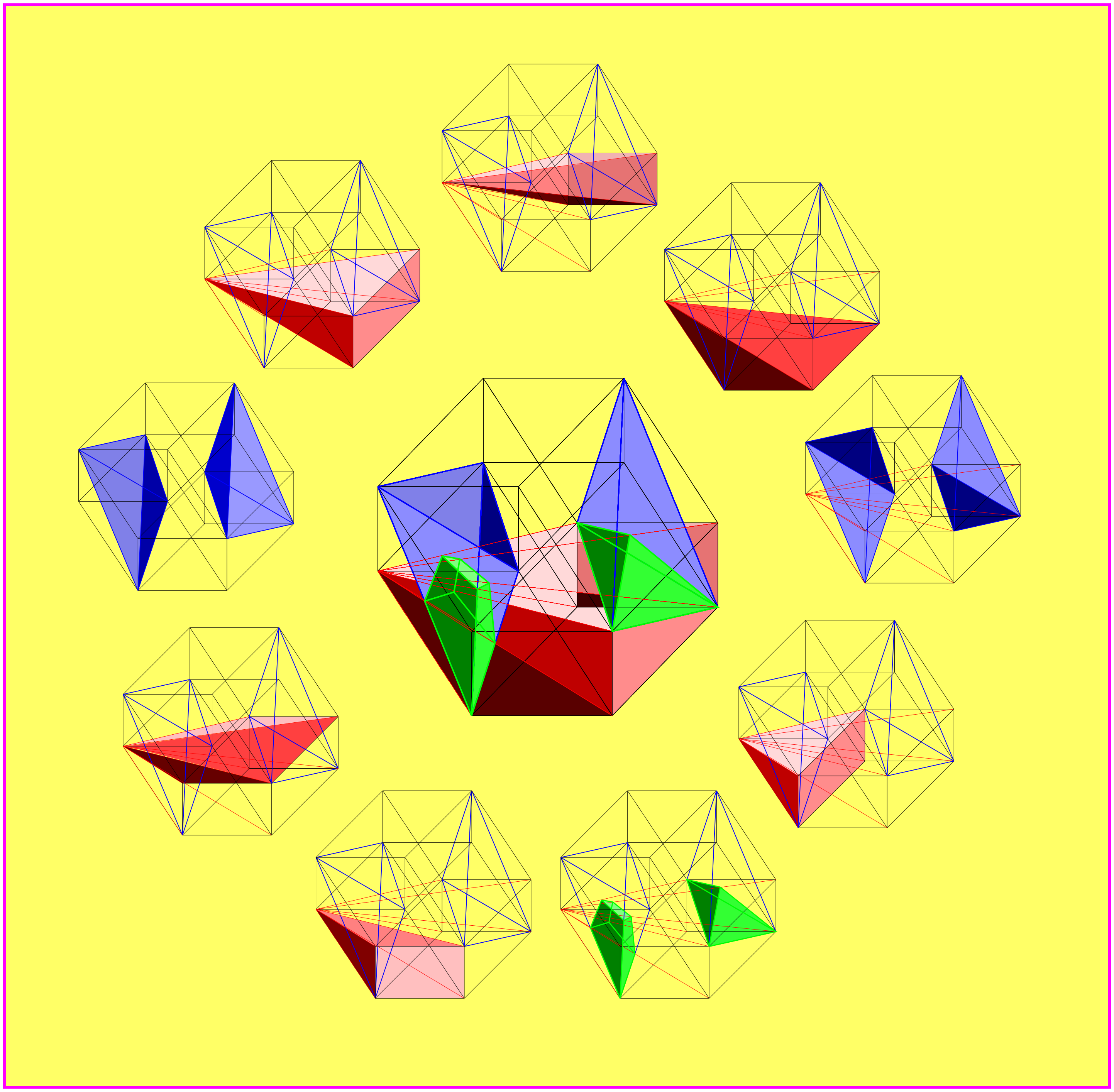


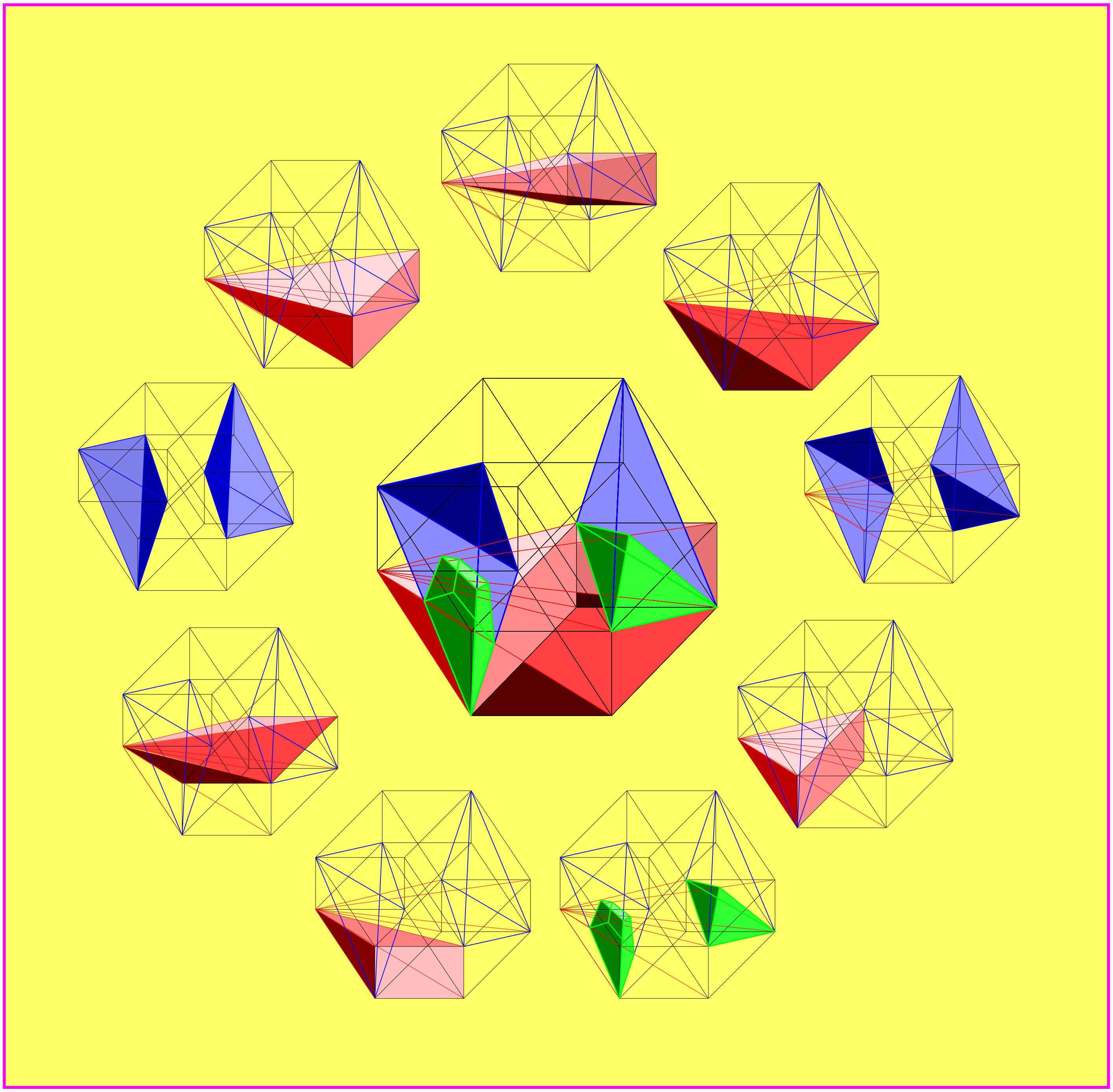




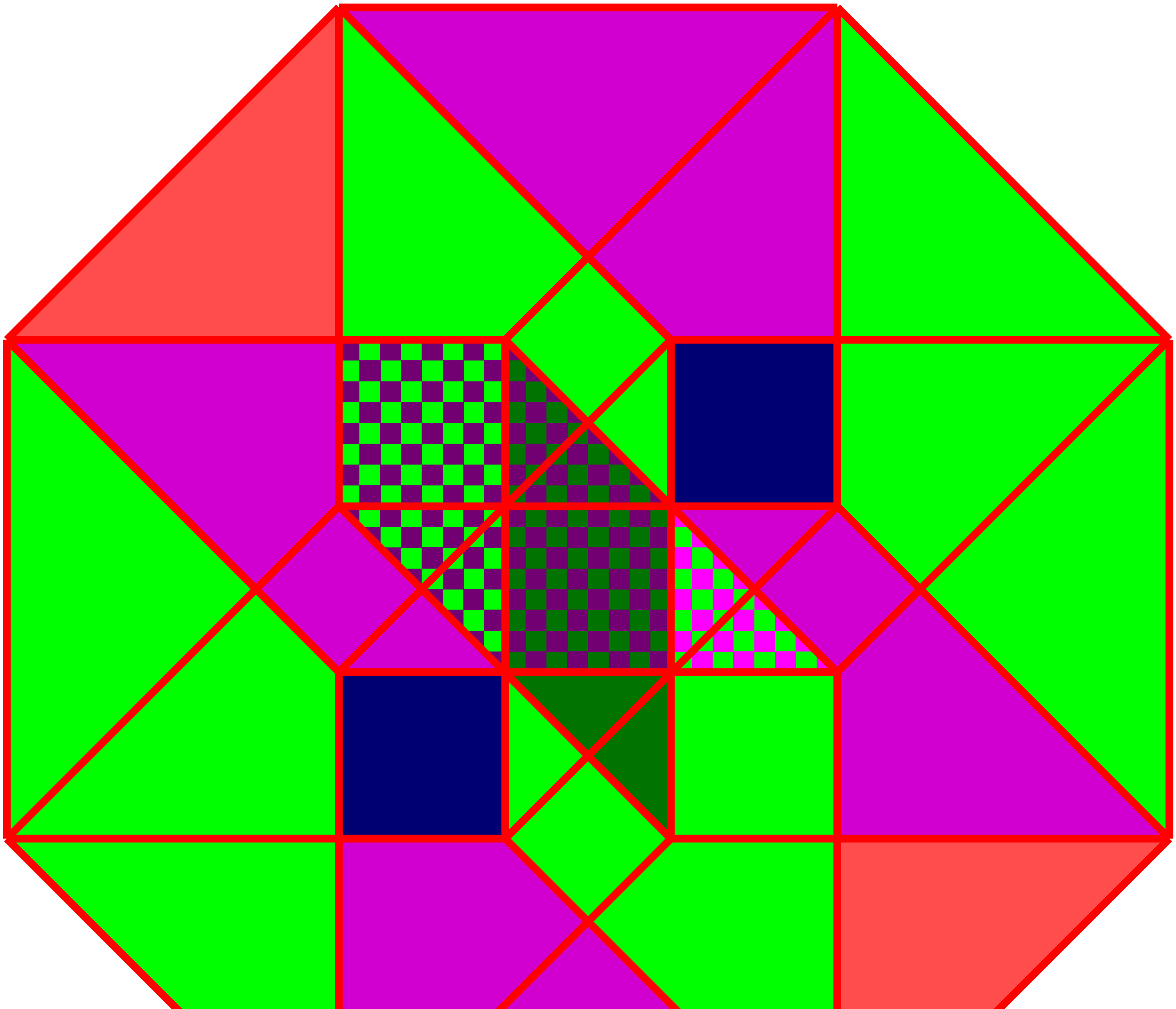


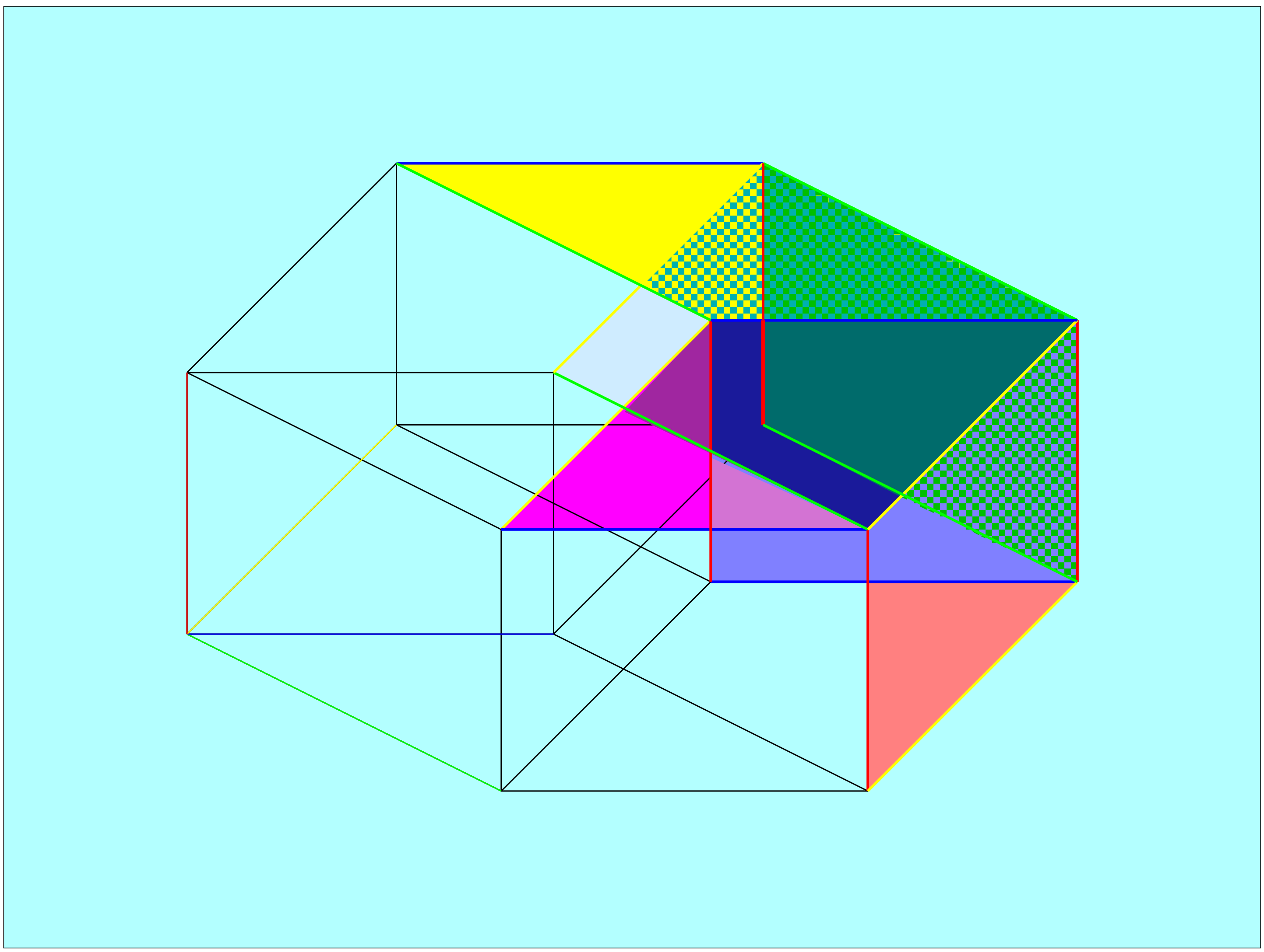


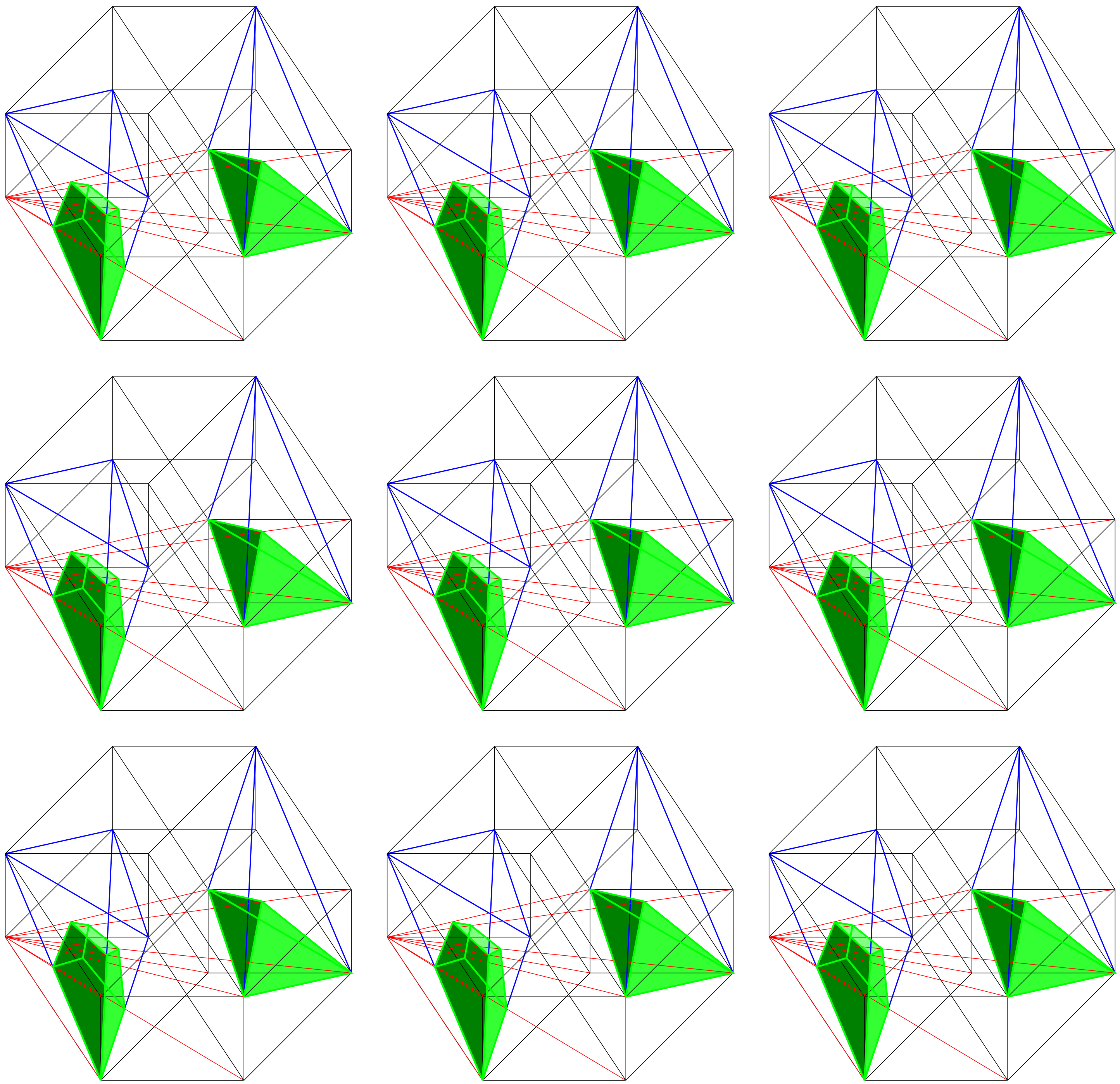


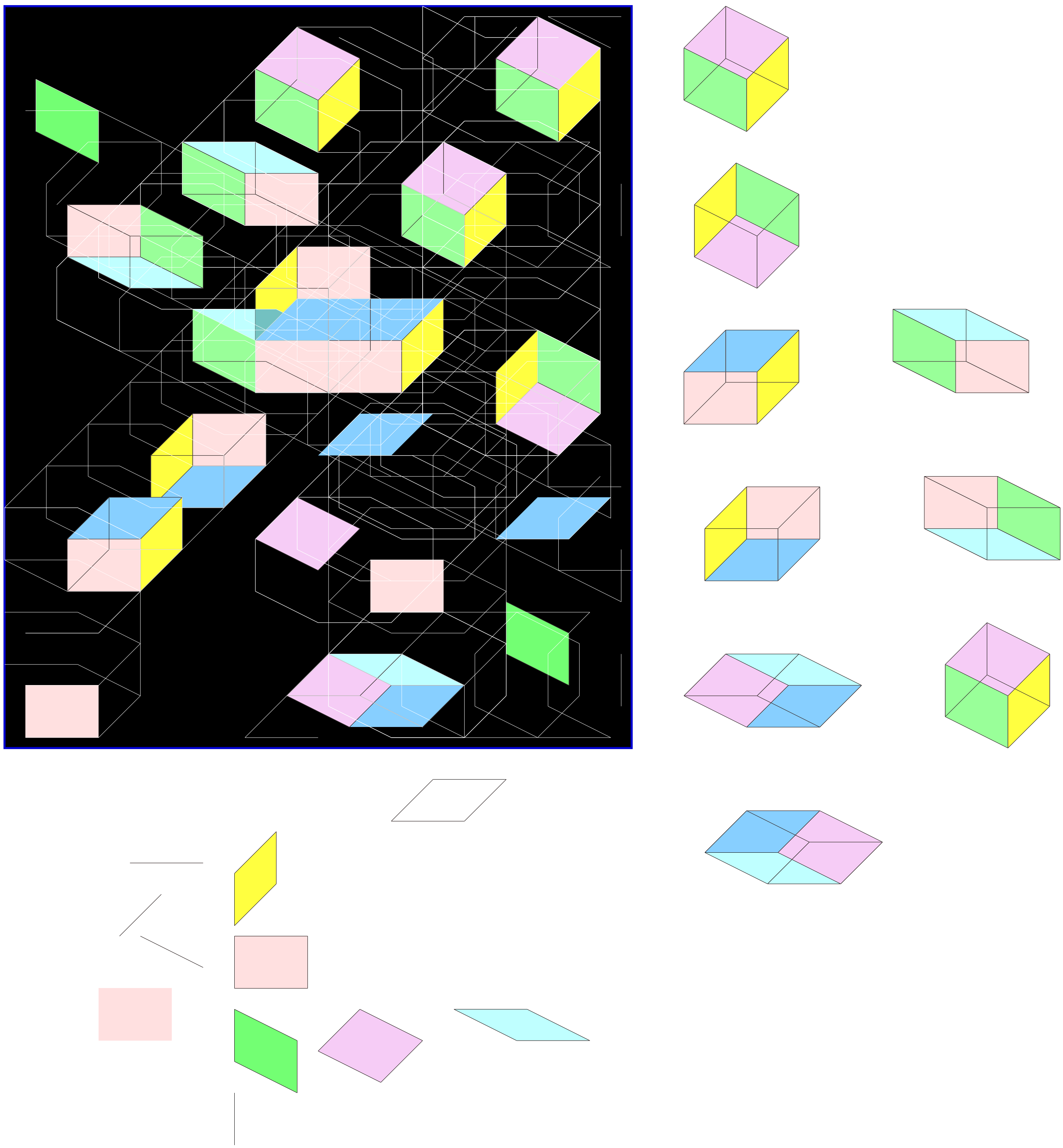


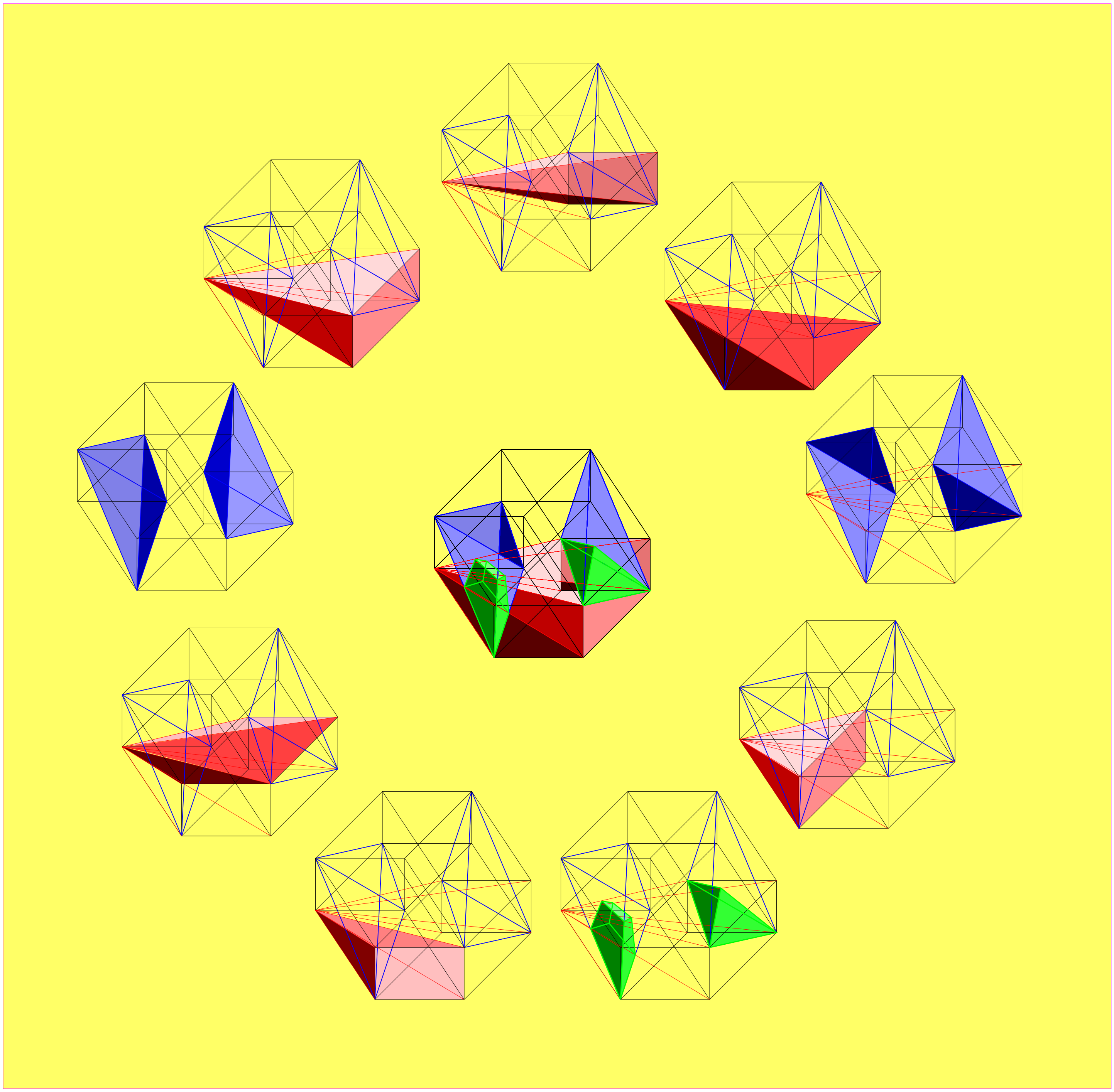
Inside a 4-dimensional cube, the 3-dimensional top cell (x,y,z,1) is coned to the origin, (0,0,0,0). The red pyramids indicate the six pyramidal cells on the various 2-dimensional faces of that cube. The blue tetrahedron on the left is the smallest convex set that contains the unit coordinate vectors. The The blue tetrahedron on the right is the smallest convex set that contains the four points (1,1,1,0), (1,1,0,1), (1,0,1,1), and (0,1,1,1). The red cone intersects these two tetrahedra in green solids. On the left is an object that has 2-dimensional faces which are kites and which together form a cube-like object. On the right, four triangular faces form a tetrahedron. Either of these figures can be rotated and reflected within the 4-cube, so that four copies of the green objects tile the blue tetrahedron. Under such rigid motions, the cubical base of the red pyramid rotates to (x,y,1,w) then to (x,1,z,w) and finally to (1,y,z,w). Meanwhile, the resulting four copies of the 4-dimensional pyramid fill the 4-cube. Thus the original cone occupies a quarter of the space in the 4-cube. Therefore, the sum of the volumes of cubes whose edge lengths range from 0 to 1 is 1/4. Analogous phenomena occur in all dimensions.

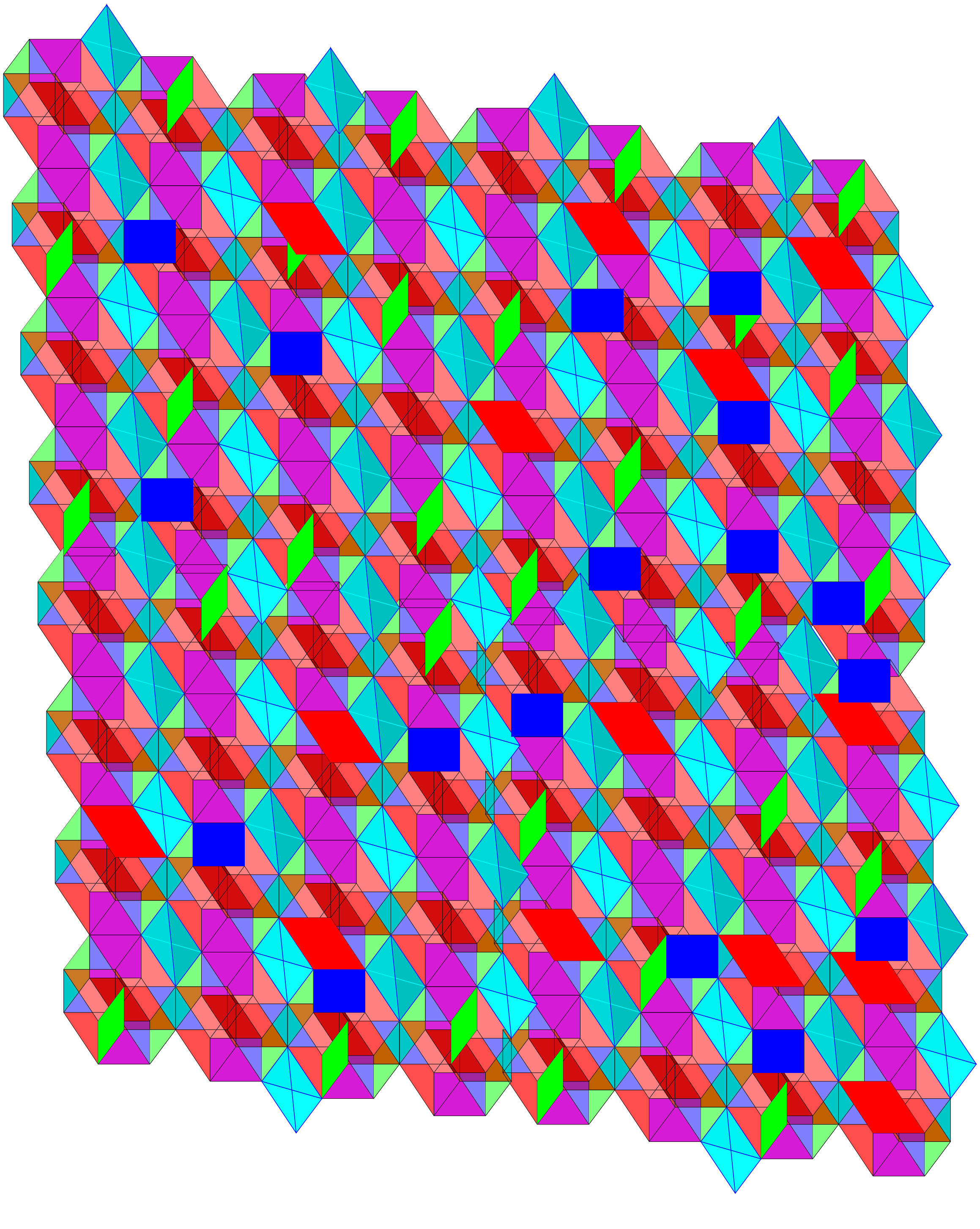


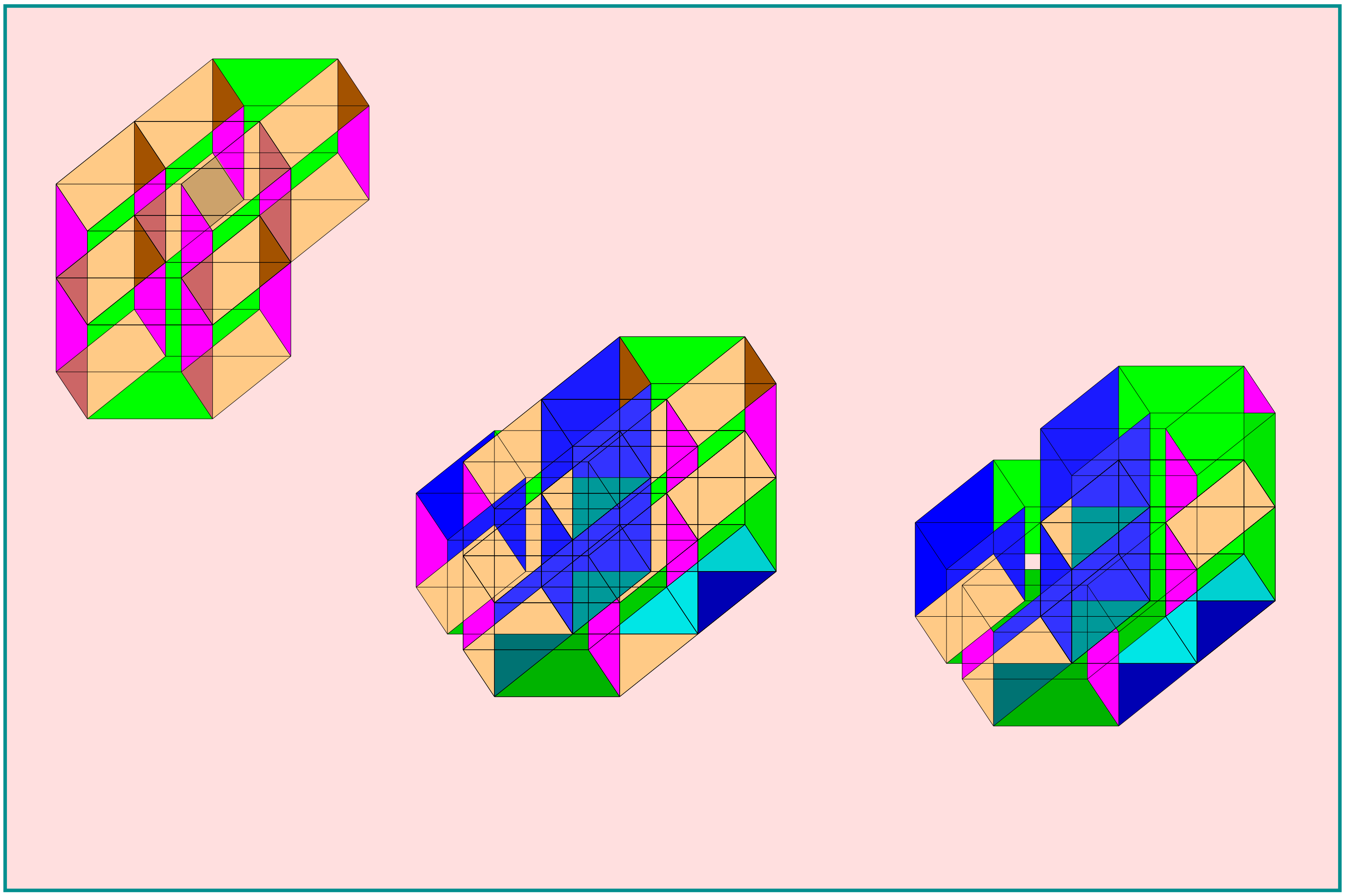


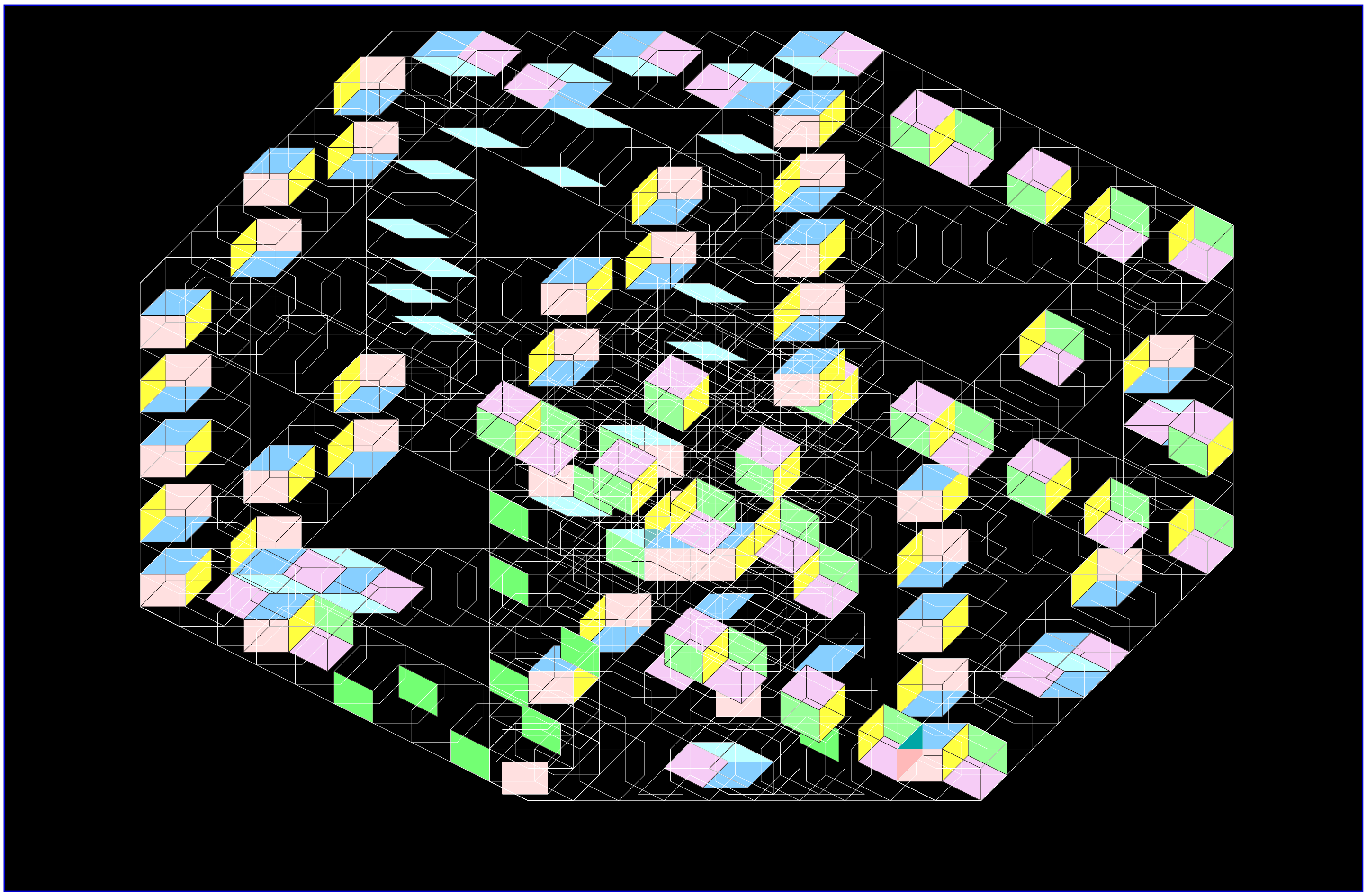


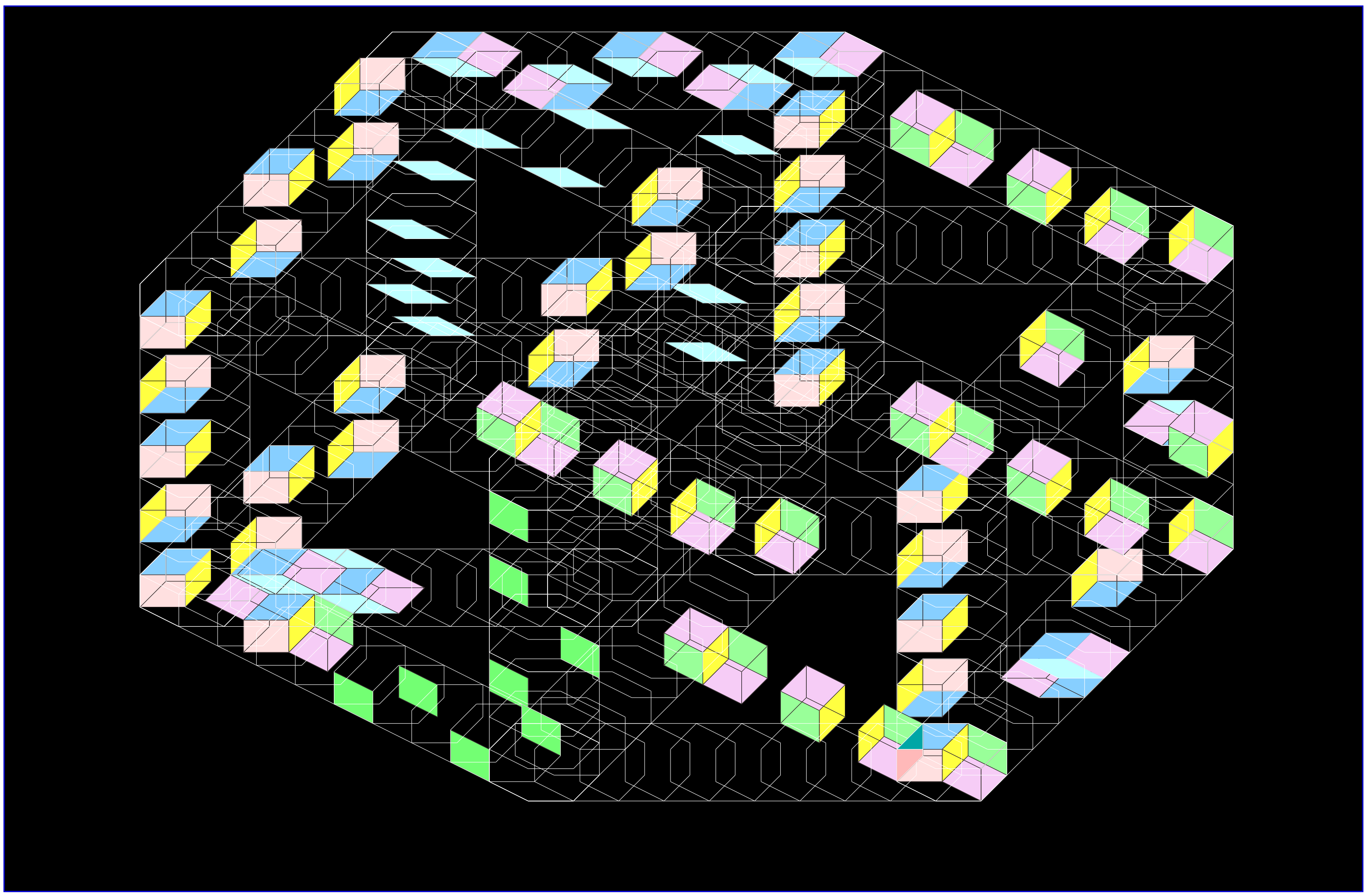


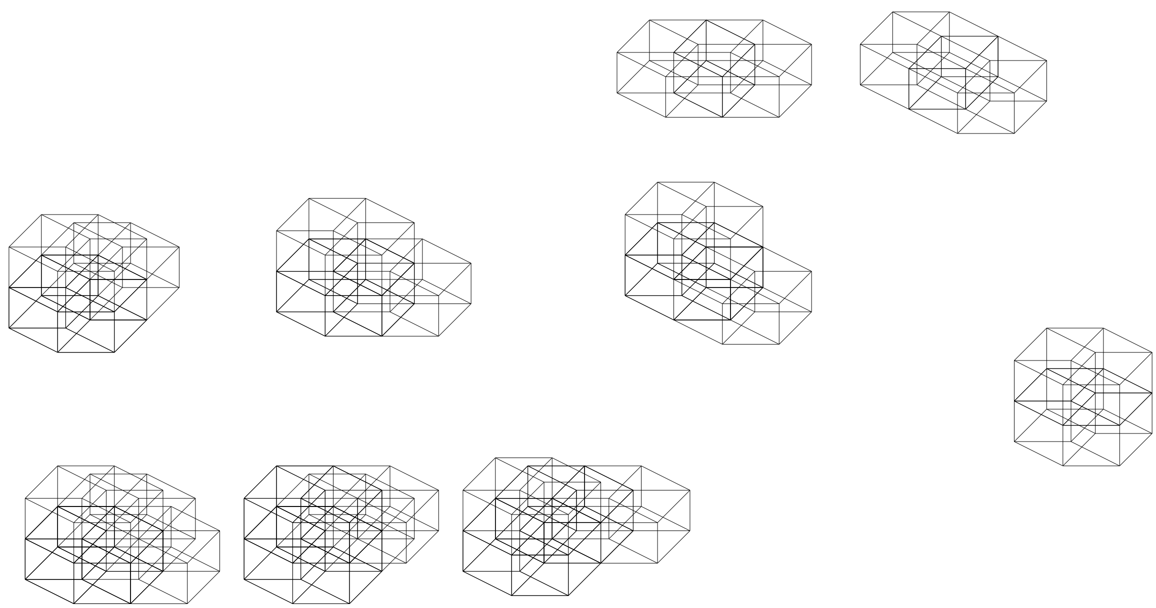












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