Fractals in Higher Dimensions

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University of South Alabama, or anyone else of whom I can think.

Dedicated to

My Family
&
My Past and Future Students

Contents

Chapter 1

This Is My Story

This book grew out of a pair of science fair projects that Albert, my son, wrote several years ago under my direction. I am a professional mathematician who knows very little about what is being written here. I actually learned this to explain it to Albert who knows even less since he never found the time or the inclination to read the things that I have written here. Such is life. We develop ideas in such a way that they should be accessible, but the access to math is by doing it and discovering it yourself. Albert has been too busy. It takes a long time to learn this lesson: to learn anything you have to try.

That is not to say that Albert is not a smart young man. He is. Mathematics did not catch his fancy the way I thought it might. I think he still enjoys it and appreciates it, but he is more interested in the social sciences. We give our children opportunities, and allow them to make choices for themselves. Albert's choices are his. My own choices were not what my own dad would have chosen, and his choices were different than his father's were. Thank goodness for variety.

Within American society today very few people choose to do mathematics. As I was growing up, scientific and technological progress was integral to the social experience of young people. The space race and the cold war had a lot to do with that. The Russians might have beaten us. In a certain sense they did. The Russian scientists who have since immigrated to America are, by and large, better educated that the average American scientist. Society does not care not since the west won the space race, and we won the cold war. At the time of this writing the next big scientific challenges include developing renewable energy sources and developing a method to reduce global warming while maintaining our industrial standard of living. These challenges are by no means trivial, but I am sure that mathematical intuition, and mathematical techniques will play a central role in these problems. Currently developments in the biological sciences include mathematical methodology. So from my point of view, mathematics permeates the problems that we need to solve.

I think that many people do not choose to study mathematics because they think that one has to have natural talent in order to do mathematics. I am not sure what natural talent is. Here is my theory. If you think you have natural talent in some aspect such as music, sports, car repair, or something else in which you excel, think back. Your parents, friends, or teachers probably taught you something at an early enough age that now it seems simple. Your initial advantage was only marginally more than your sister, brother, or next door neighbor. Someone in your life helped foster your talent. Now there are some things that you can do better than most others.

By reading this book carefully, and working through the ideas here, you can develop some talent in mathematics. But the true fun in mathematics is not in reading the works of others, but in taking that work and making your own mathematics from it. Certainly one of the reasons that I am writing this is so you will take the ideas and make them work in other places.

Doing mathematics can be difficult. Many things in life are. Language can be difficult, yet most of us acquire it through practice and because there is a need to communicate. We don't think of the subtleties until we try and learn a different language.

Mathematics is a language, too. It is necessary to communicate, and some anthropologists believe that mathematical communication (accountability) specifically motivated written language. Oral traditions, after all, have been used to preserve stories and poems through out history — even before language was written.

Many people do not feel the need to communicate the intricacies of mathematics beyond the basic arithmetic that affects their jobs. People who claim, "I was never very good at math," often have to be involved in arithmetical computations (such as an estimate of a car payment or their own sales commissions for the month) that would make my professional hair stand on end. These people may not be good at seeing that subtle algebraic and symbolic manipulations have an intrinsic beauty that transcends their applicability. Learning to look for the intrinsic beauty is the hard part because the intrinsic beauty is only revealed by the practice of mathematical thought. It is as music that only the performer can here.

Don't get me wrong, applications of mathematics are great, but when we mathematicians pursue mathematical beauty for its own sake we often find applications beyond our original horizons. Facts about functions between higher dimensional disks have applications akin to Nash's theorem — the easy one, the one that earned him a Nobel Prize. The fictionalized account is found in the movie, "A Beautiful Mind." A great scene in that movie is when he shows the girl the umbrella among the stars. Mathematicians often pick out patterns from among seemingly chaotic patterns. This last sentence would be a great segue to the story that I will develop here, but I have a few more things to say first.

Mathematical beauty is the motivating force for developing this book. Upon the first development of the principal movie, I asked a colleague, "What is it good for?" The reply was, "Well, it is very beautiful." This was for me reason enough to continue developing the work.

When we pursue mathematical beauty for its own sake, it is less dangerous than pursuing physical beauty for its own sake. The latter fades, the former can be seen in many different lights and throughout time. One of the most beautiful books ever written (which is motivated by applications) is Euclid's "Elements." It is as good today as it was then. In trying to understand its subtle beauty, many fine mathematicians discovered the even more applicable ideas of hyperbolic geometry. Hilbert axiomatized geometry more fully and articulated those axioms which were tacit in Euclid. He asked whether all of mathematics can be axiomatized, and as a result of these questions, the modern digital computer was born. So again, with beauty as the primary mover, applications flourished.

There are some romantic notions in mathematics. The trigonometry functions are found among the earliest. Matrices, rotations, the binomial theorem, characteristic functions, higher dimensional figures, figures whose dimension is fractional, chaos, the uncountability of subsets of the real numbers, the real numbers themselves, triangles, tetrahedra, platonic figures in all dimensions, symmetry: they are all here. I even use limits (which are found within the suzerain of calculus) to inform one of the proofs here.

As you may know from reading my previous works, I am a topologist; the ideas developed herein are not central to my mathematical research. From a professional's point of view, there will be ideas that are lacking, or an elegance of form that is missing. The development may seem to be that of an awkward amateur. My defense is simple. It is the work of an awkward amateur. The world needs more professional books about mathematics for the lay person. I have some experience in writing books, so I thought I would give this one a try. I did leave the work incomplete for some time. I think I was not sure that I could complete it. As I write this sentence, I have made a commitment to myself to finish it. Only when it is published will we know if my commitment was met.

The ideas that are developed here are among the tools that are developed in an undergraduate education in mathematics. As a teacher, I hope that every math major that I have had the pleasure of knowing, would be able to pick up this book (indeed the contents of the next chapter) and understand all of the ideas with a little bit of work. These are among the things that you should know. They are among the tools that I employ without much thought. Yes, you should also know the concepts of topology and abstract algebra, but they don't enter into this book in a big way. Except that when topologists build our usual array of topological spaces, we often use n-dimensional simplices as the bricks of our spaces. Our simplices are symmetric, and symmetry is well within the suzerain of abstract algebra. So there are background ideas here that may be useful (applicable?). And when we build exotic spaces, we often use a Cantor-like construction. The sets we are illustrating are akin to Cantor sets; for certain parameters they are Cantor sets, but I do not prove that here.

This book a romantic piece of non-fiction that was inspired by the famous romance of many dimensions: "Flatland" and its descendants. I am privileged to count among my acquaintances the artist Tony Robbin and the mathematician Tom Banchoff. Tom was one of the first people to actively develop computer graphics so that we could begin to see things from the fourth dimension. He claims to have been influenced by comic books. Tony was influenced by Tom. He has developed a method of painting and illustrating objects from higher dimensions with an artist's aesthetic. After meeting Tom, Tony wrote his own computer program to illustrate the hyper-cube. I saw one of Tony's works in a gallery, and met him ten years later. The art spoke directly to me, and my admiration developed as consequence. Tom has written a nice book called, "Beyond the 3rd Dimension," which is an excellent primer on higher dimensional thought that is written for the layman. Tony also has written extensively about the 4th dimension. His book "Fourfield" describes how one can and should stop thinking of the space-time continuum as a series of 3-dimensional moments and start thinking of it as a 4-dimensional whole. His most recent book "Shadows of Reality" articulates these ideas further. In admiration of these gentlemen's works, the book you are reading now is engendered.

However, my point of view is different. I don't want to hide the math. Tom didn't either, but he indicated a lot of the math through pictures of the real world, and his book didn't have many equations. I want you to see what is involved, so that you can take these ideas and improve upon them. My romantic notion is that I will affect you to pursue this path of beauty.

Many physicists write for the lay public and remove equations from their books. At least one describes his equation in words. It takes the educated reader a few minutes to transcribe the words into equations. I don't particularly like story problems! I would rather see the equation lying naked and beautiful before me. Well, the unwrapping of a story problem is part of the pursuit, but I still get my satisfaction when the mathematics unfolds.

Let me comment upon the sexual innuendos in the preceding paragraph. Explicit and implicit images permeate our society. Sex has been used in advertising since the medium was born, and it will continue to be used. Violent images, too, are a part of the media; these may be more dangerous. Explicit mathematics however is taboo.

We are conditioned to hate math, to think that math is too difficult for anyone but a brainiac, and to think that mathematicians are a nerdy lot with no social graces. The truth about mathematicians is that they are less likely to act in a judgemental fashion about their colleagues race, sex, country of origin, or lack of social graces than other segments of societies. I propose that by including the scientific and the mathematical and integrating these into the fabric of society, we will develop a more healthy, and less prejudicial social culture. My romantic notion is that mathematics, logical thought, and scientific inquiry can help humanize us. Clearly, the arts also have a role in the humanization process, but society has ignored the essential role that mathematics and science can play.

There are so many soap boxes to stand upon, and the story has yet to unfold. What does Albert have to do with it? Albert is the "he" in the following paragraphs. I gave Albert a couple of projects to consider for science fair. The first of these was to play the chaos game in regular polygons using the n-ary expansions of certain irrational numbers as generating sequences for the patterns. The second was of his own design. He used the idea of the chaos game to play in the five platonic solids. These are the tetrahedron, the octahedron, the cube, the dodecahedron, and the icosahedron. His project involved developing flip-book animations of various cross-sections and projections of these figures. One projection of the chaos game in the tetrahedron looked a lot like the chaos game in the square. The latter is a homogenous figure since the $1/2$ rule evenly distributes points within the square. This led us both to think that the figures in the polygons were projections of figures from higher dimensional spaces. Indeed they are.

Specifically, the n-dimensional simplex, which is defined in detail below, projects to a regular polygon with $(n+1)$ vertices. The chaos game when played within the higher dimensional set, projects to the figures in the polygons. The figures in the polygons are well known to those who play the chaos game professionally. There are many web sites devoted to the subject. Less seems to be known about the higher dimensional aspects of the theory. So I developed this introductory opus to induce you to become engaged in a discussion on the subject. There is a lot more work to be done within this context. But I think I have done a good job of introducing the subject. Please develop it further.

Here is the plan for the book. Chapter 2 lays the mathematical framework for the computer programs that generate the images. In Chapter 2, we describe the n-simplex, the chaos game, the Sierpinski sets, matrix rotations and projections, and the multinomial coefficients. Chapter 3 contains the text of our computer programs. These are replete with comments that refer back to Chapter 2 and provide a guide to the programming lines. Since I am writing in MATHEMATICA, the code is very self-explanatory. However, there is a slight annoyance with MATHEMATICA. Sometimes I want to use a word to describe a function, but that word is a MATHEMATICA command. MATHEMATICA commands always start with a capital letter. But if I try to name function with a lower case letter of the same name, I receive a spelling warning. Such warnings can be helpful, but to avoid them I often rename the variable with a rhyming word. For example, if I want to use the word “choose” to avoid a spelling conflict, I use the word “booze.” The margins contain images from modified programs. The full set of images is found on the accompanying disk in the form of animated movies.

As I write this paragraph, I hope that I am in the final stages of my editing. One thing that I am paying particular attention to is my choice of pronouns. By “we” I usually mean you and I, but occasionally I mean we mathematicians. I personalize the discussion with “I” rather than a regal “we” when a new idea is introduced. The choice of “we” to mean “you and I” usually comes after an idea has been introduced or as a concept is becoming more clear so that the concept is a shared idea between you and me. “You” always refers to the current reader. When I use “we” to mean mathematicians, I mean that the terminology or cultural reference is standard. Only a mathematician would bother to explain that!

I would like to take a few moments to thank everyone who helped me during the production of this book. My son Albert read several sections and improved the exposition at many key places. My colleagues and students have been very patient with me as I indicated the imagery to them. My other two sons, Alexander and Sean have also responded with “cool” as I showed them the images. Ed Dunne the acquisitions editor at the American Mathematical Society had a number of key suggestions on technical aspects. These made the flip-book portion work. He and I had a very lively discussion in Pittsburgh, PA about what could and couldn’t be in a mathematics book. He and I were in agreement about making mathematical discussions more lively. Shortly, after the Pittsburgh meeting my father died. I hadn’t really thought about that until editing this paragraph. I think that there was a delay in development because of that association. Ed has been fairly patient with me, but as all good editors, he has insisted. So I am working. Finally, I want to thank my wife Huong, who was very understanding at several times as I was working on this book. It took longer than I expected. I hope it is better because of the delay.

Chapter 2

Gratuitously Explicit Mathematics

1. Preliminaries

Mathematics does not have the luxury that fiction does. In fictional writing, the author can allude to an event without telling the reader exactly what happened. The reader then infers the action. A mathematics text must give the reader enough details so that the reader can more or less understand the concept. Certainly, the best mathematics texts leave details out and state notions in the form of exercises so that the reader can workout for herself important fundamental ideas, and in some cases I do so here. But in every case, the fundamental definitions must be formulated precisely. A novel that takes place in a city need not specify the city unless the city's geography plays a role in the action or in the author's mind.

Many of the ideas and notions that are presented here are like the familiar landscapes in a city with which an author is familiar. But even though I may find them familiar, I will draw you a map and provide a clear set of directions to help you find your way through. Although you may get lost, reading further, trying an exercise, and glancing back will usually clarify a concept. Don't worry about not understanding an idea the first time you see it, or the second, or the third. If you work with the idea within the broader context, it will eventually come to you. With this method, that realization will not necessarily occur because of anything that I have written. Instead, it will come to you through your own thought process.

This chapter contains a lot of mathematics, and I have made attempts to make it accessible to a student who has learned through trigonometry, by introducing definitions within the context of the book. Sometimes an idea is introduced informally and the formal definition is given within a few paragraphs. This chapter is replete with notation. A full glossary appears as a separate chapter after the computer code and before the index.

I use the notation $\mathbb{R}$ to denote the set of real numbers. These include positive an negative numbers, fractions, radicals, π, and an infinity of other numbers that do not have simple descriptions. This infinitude is contained in the set of infinite non-repeating decimals. I will work with the real numbers as if they are familiar. The set of *real numbers* is identified with the set of points on a line that extends infinitely in both directions; it is the set of directed measurements of arbitrary precision. The coordinate plane is denoted by $\mathbb{R}^2$. This is the set of ordered pairs of real numbers. More generally, $\mathbb{R}^n$ denotes the set of ordered n-tuples of real numbers where n is a positive whole number such as $1, 2, 3, \ldots$ (The three dots form an *ellipsis*, it is read "and so on"). Such an ordered n-tuple is written as $(x_1, x_2, \ldots, x_n)$ where each x_j is a real number for $j = 1, 2, \ldots, n$. I will attach geometric significance to the set $\mathbb{R}^n$ for $n > 3$ as I develop the text. On the other hand, you should be familiar with 3-dimensional space, $\mathbb{R}^3$ since the world appears to be 3-dimensional. You may not yet know how to measure distances and to give coordinates to points, but the space should be familiar to your senses. I use the term *n-dimensional space* or *n-space* to mean the set $\mathbb{R}^n$.

Many of the sets that we consider are *n-dimensional subsets* of $\mathbb{R}^{n+1}$. That means they are like higher dimensional skim-boards. A skim-board is a thin piece of wood or plastic that skims across the surface of the water. The water under the board is inherently 3-dimensional. The board appears to be 2-dimensional, but it does not encompass the totality of the surface of the water. The skim-board is kept on plane via the distribution of the weight of the skimmer. The subsets that we consider are strictly smaller than the space in which they sit. The number of degrees of freedom in the subset is one less than that in the bigger space. Paint covers a skim board, and the ocean is full of fish. Paint forms an essentially 1-dimensional covering of an essentially 2-dimensional object. Fish are essentially points in the space of the sea. Well, maybe flounders are 2-dimensional inhabitants at the bottom of the bay. Mathematical examples of sets with one fewer degrees of freedom consists of lines in planes and planes in space, or more specifically, segments of lines in planes and triangles in space.

In general, we will look at subsets of $\mathbb{R}^{n+1}$ and use set builder notation to denote such sets. The notation

$$\{x \in S : \text{Property} \quad P(x) \quad \text{holds}\}$$

is read "*the set of all x in S such that Property P of x holds.*" The funny character "$\in$" is read to mean "in" or "is an element of." The curly braces "{" and "}" denote where the set begins and ends, respectively. (This notation is used in a different context in the computer programs). The colon ":" reads "such that." The notation means that we are looking at some elements of a set whose name is S. Those elements are determined by some equation, inequality, or property that we symbolize as $P(x)$. The preposition "of" sometimes means multiplication in a mathematical context; here it means that the property, equation, or inequality depends on x. Similarly when a function is specified by a formula one often writes $y = f(x)$ to indicate that the set of y values depend on a variable x via a formula f, and the equation reads, "y equals f of x."

This chapter is organized as follows. First, the notion of the n-simplex as a subset of $(n+1)$-dimensional spaceis developed. This development proceeds through the lower dimensions up through higher dimensions. All of the constructions of the computer program occur in the n-simplex. Next I describe the chaos game. I indicate how the chaos game in the triangle simulates Sierpinski's triangle. The set of points in the n-simplex that are reached by an infinite chaos game approximates a set that is related to a Cantor set. So each Sierpinski subset of the n-simplex is described by a Cantor-like construction. The next section deals with rotation and projection. I rotate the constructions in higher dimensions and project the result onto the plane of the paper. In doing so, I hope that enough facets of the figures are illustrated for you to view the intrinsic beauty of these sets. Also in this section, I develop a process of projecting and unfolding pictures that constitute one of the marginal flip-books. The final section of this chapter creates a correspondence between a black and white version of the pictures and the combinatoric structures that are encoded in the multinomial coefficients. These coefficients are obtained by expanding the nth power, $(x_1 + x_2 + \cdots + x_r)^n$ and grouping like terms.

It is time for the show to begin.

2. The n-simplex

This section is arranged as follows. The 1, 2, and 3-simplices are defined as subsets of the visible dimensions. Then I define the n-simplex in general. If the first bits seem too boring, feel free to skip ahead to the general case and return to specific cases as needed.

2.1. A Line Segment. Consider the set of points in the (x, y)-plane such that both the coordinates are non-negative and the two coordinates add up to equal 1 (Fig. 1). This is written as the set $S_1 = \{(x, y) \in \mathbb{R}^2 : \; x + y = 1, \; \& \; x, y \geq 0\}$. It is a line segment in the first quadrant whose slope is -1. This segment will also be called the 1-*simplex*. As you might imagine from its name, it is the first set among a sequence of infinitely many that we will examine together. In a moment, I will define the 2-simplex, the 3-simplex, and so forth. Before I do, let me tell you more about this line segment.

It is one diagonal of the square whose vertices are $(0, 0)$, $(1, 0)$, $(0, 1)$, and $(1, 1)$. As the diagonal of a *unit square* (square with area

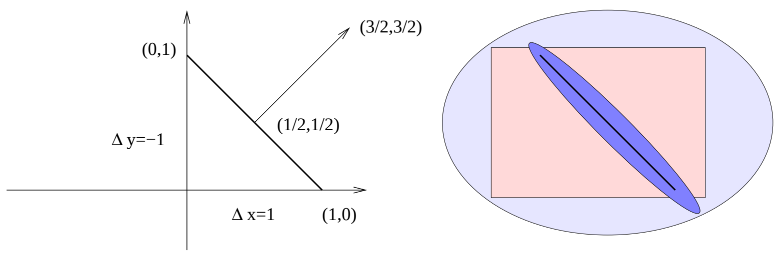


Figure 1. A line segment, its perpendicular, and some convex sets that contain it

1), the length of this segment is $\sqrt{2}$. Its end points are $(1,0)$ and $(0,1)$, and the distance between these points is given by the formula (Fig. 3).

$$\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$
$$= \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{1+1} = \sqrt{2}.$$

The segment is the intersection of all convex sets that contain these two points. A set is *convex* if for any two points in the set, the line segment joining these points is also in the set. Such a line segment is convex by default (Fig. 2).

The notation Δx is commonly used to denote the change in x. To measure that change, take the difference between the two x-coordinates of the given points. By the way, the *slope* is measured as the ratio in the change in y to the change in x. That is the slope is measured as

$$\frac{\Delta y}{\Delta x} = \frac{(y_1 - y_0)}{(x_1 - x_0)} = \frac{(0-1)}{(1-0)} = -1.$$

Here and above, the endpoints of the segment have been denoted by $(x_1, y_1) = (1,0)$ and $(x_0, y_0) = (0,1)$.

A concept that is dual to the slope of a line is a direction that is perpendicular to the line. Such a direction could be up from or down from the line, and it can be determined by an arrow in the plane that points perpendicular to the line. Such an arrow could point from $(0,0)$ to $(1,1)$, or it could be any arrow parallel to this. Ordinarily we illustrate such a parallel arrow as pointing from $(\frac{1}{2}, \frac{1}{2})$ to a point such as $(\frac{3}{2}, \frac{3}{2})$. The idea is that the arrow starts at the center of the

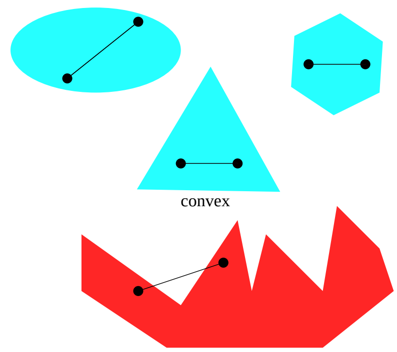


Figure 2. Defining Convexity

line and points upwards. The reason that such a concept is needed is that higher dimensional flat sets (higher dimensional skim boards) do not have slopes, but they do have perpendicular directions. Such a perpendicular could be rescaled to be of unit length, but we don't need to do that here. You can if you want; I won't. A higher dimensional surfer, stands on the skim board so that her center of gravity is on a line perpendicular to and above the center of the board.

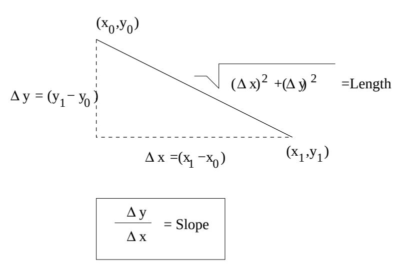


Figure 3. Slope and distance

There are a few more things that I should say at this point. First, most of the preceding paragraphs of this section should be a review of those things that you were exposed to in your first course in algebra. I imagine that you may have forgotten a lot if this material, and therefore I have reviewed it. The words describe the picture that has been illustrated, and the words are necessary since I am going to develop the concept into dimensions into which we cannot see. My job here is to bore you with the familiar and then generalize it until it seems uncomprehensible. At that stage, your job is to reason through the material by going back to the mundane situation until you see the analogy. In the next several sections, I will continue to give detailed descriptions of familiar sets so that when you meet the unfamiliar you will be prepared via the lower dimensional analogues.

Mathematical induction is the process by which we generalize an idea from a few selected cases to the full set of positive integers (whole numbers). The integers themselves can only be defined inductively. That is, every positive integer has a successor. The process goes something like this. Having defined a concept in an initial case, we show that the concept can be generalized from any case to its successor. Thus a mathematical theory can be built brick-by-brick as follows. We lay the first brick, and we show that if the nth brick is in place, then we can lay the $(n+1)$st brick. Mathematical induction is at the heart of all of the discussions below, but I adopt a more gentle approach. In this section, I described the first brick in detail; in the next two sections I will describe the second and third bricks in as much detail as I did here. In the fourth section, I develop the full abstraction and then specify the situation to the almost visible 4-dimensional world.

Mathematical induction is an infinite sequence of dominos. We know that if the nth domino falls, then the $(n+1)$st domino will fall. But that is not enough to know that all the dominos will fall, we must also know that the first domino falls.

An alternative view of mathematical induction comes from the cartoon character Olive Oil who slurps an infinitely long spaghetti

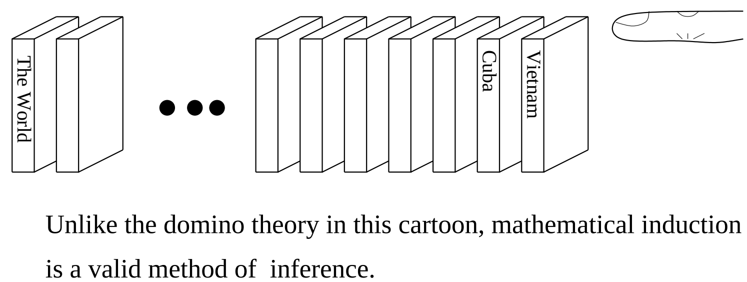


Figure 4. A mathematical inductive sequence of dominos

noodle. Once the noodle has started to be eaten, Olive is obliged to continue eating. There are three parts of the noodle: That which has been consumed, that which is at the pucker of Olive Oil's lips, and that which outside her mouth. If there is spaghetti at her lips, Olive Oil must continue slurping the noodle. She takes a bite, and there is spaghetti at her lips. Therefore she must consume the infinitely long spaghetti noodle. Mathematical induction allows Olive Oil to continue indefinitely. Won't Popeye be happy.

2.2. An Equilateral Triangle. In 3-dimensional space, consider the set of points in which all three coordinates are non-negative, and these three coordinates sum to 1. A 3-dimensional space is a space much like the room in which you are sitting. You can think of the floor as the (x, y)-plane, the north wall as the (y, z)-plane, and the west wall as the (x, z)-plane. As it happens, I am sitting here facing north and the west wall is to my left. Of course, you may not even be inside, or the room in which you are sitting may have round walls. However, I am reasonably sure that the place in which you find yourself at this moment appears to be 3-dimensional. Whether it is or not is a matter for physicists to decide.

Now let us return to the set in question:

$$S_2 = \{(x, y, z) \in \mathbb{R}^3 : \ x + y + z = 1, \ x, y, z \geq 0\}.$$

It contains the three points $(x_1, y_1, z_1) = (1, 0, 0)$, $(x_2, y_2, z_2) = (0, 1, 0)$, and $(x_3, y_3, z_3) = (0, 0, 1)$. Indeed the set S_2 is an equilateral triangle with these three points as its vertices. The edges lie in the coordinate planes; when $z = 0$, we can identify this edge with the segment of the preceding section. The length of each edge is $\sqrt{2}$ as it was above since one of the coordinates is constantly 0 along each edge. Figure 5 indicates the set under consideration.

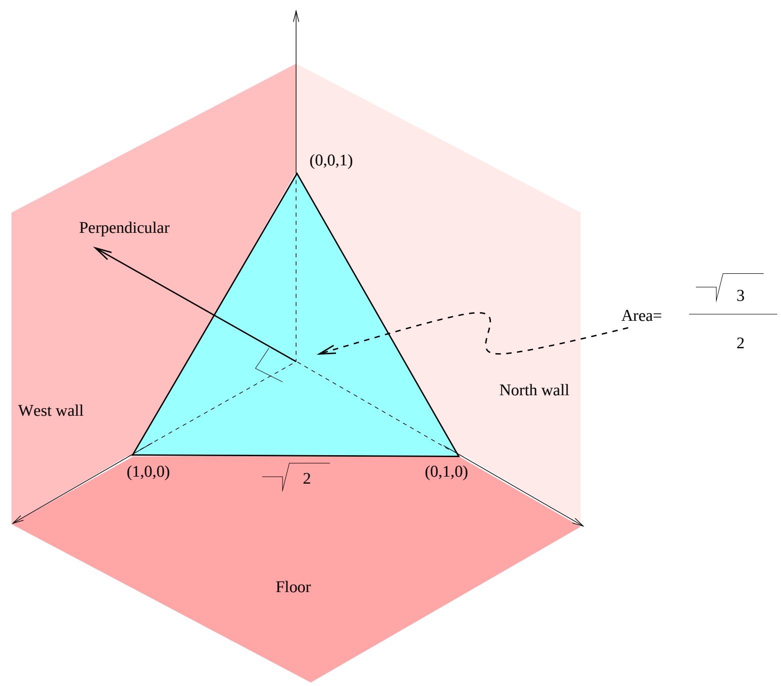


Figure 5. An equilateral triangle in the positive region of space

In the room analogue, you may think of the point $(1, 0, 0)$ as lying on the intersection between the west wall and the floor, the point $(0, 1, 0)$ as lying on the intersection between the floor and the north wall, and the point $(0, 0, 1)$ as lying on the intersection between the north and west walls. For the sake of measurement, these points each may be considered to be exactly 1 meter from the corner of the room. The edges are line segments: one lies along the floor, one lies on the north wall, and one on the west wall. If we were to run dowels along these segments, then we could stretch a triangular tent in the corner of the room. The region bounded by the tent, the two walls, and the floor would form a tetrahedron with one equilateral face. The other three faces would be isosceles right triangles. Incidentally, the volume within the tent would be $1/6$ of a cubic meter.

The equilateral triangle that we are considering is the intersection of every convex set that contains the 3 points $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. Subsequently, we will call this set the 2-*simplex.*

The area of the figure can be computed to be $\frac{\sqrt{3}}{2}$ as follows. Each side has length $\sqrt{2}$. The point $(1/2,1/2,0)$ is the midpoint of the side that is opposite the vertex $(0,0,1)$. The distance between the midpoint and the vertex is

$$\sqrt{(1/2-0)^2+(1/2-0)^2+(0-1)^2} = \sqrt{1/4+1/4+1} = \sqrt{3/2}.$$

The line that joins these two points is perpendicular to the edge between $(1,0,0)$ and $(0,1,0)$. Thus the area is given by the formula area is equal to $1/2$ of the base times the height, or:

$$A = \frac{bh}{2} = \frac{\sqrt{2}\sqrt{\frac{3}{2}}}{2} = \frac{\sqrt{3}}{2}.$$

The arrow that points from $(1/3,1/3,1/3)$ to $(4/3,4/3,4/3)$ is perpendicular to this triangle.

Later we will be considering more points in higher dimensional spaces. So it is worth calling the points $e_1 = (1,0,0)$, $e_2 = (0,1,0)$, and $e_3 = (0,0,1)$. We can always think of the points $(1,0)$ and $(0,1)$ as the former two points above. In this way we have the standard inclusion of 2-dimensional space inside 3-dimensional space. In the room analogy, the 2-dimensional space is the floor. The inclusion is merely convention since there are lots of embeddings of a 2-dimensional space into 3-dimensional space. In fact, the equilateral triangle under consideration lies within the plane $x+y+z=1$. As it happens, a number of things are more convenient when we consider this triangle in 3-dimensions.

Much later, we will be considering various subsets of the triangle and its higher dimensional analogues. For example, there are the three midpoints $(1/2, 1/2, 0)$, $(1/2, 0, 1/2)$, and $(0, 1/2, 1/2)$. These form the vertices of an equilateral triangle that is one quarter the size of the original. Thus we can cut the triangle into four equilateral triangles. One of these, for example, has vertices $(1, 0, 0)$, $(1/2, 1/2, 0)$, and $(1/2, 0, 1/2)$. We can this triangle T_1. Another triangle, T_2, has vertices $(0, 1, 0)$, $(1/2, 1/2, 0)$, and $(0, 1/2, 1/2)$. and a third, T_3 has vertices $(0, 0, 1)$, $(1/2, 0, 1/2)$, $(0, 1/2, 1/2)$. We will examine the midpoints of each of the edges of the resulting triangles and break each into four again. This process can continue *ad infinitum*. The resulting set is indicated in Fig. 6.

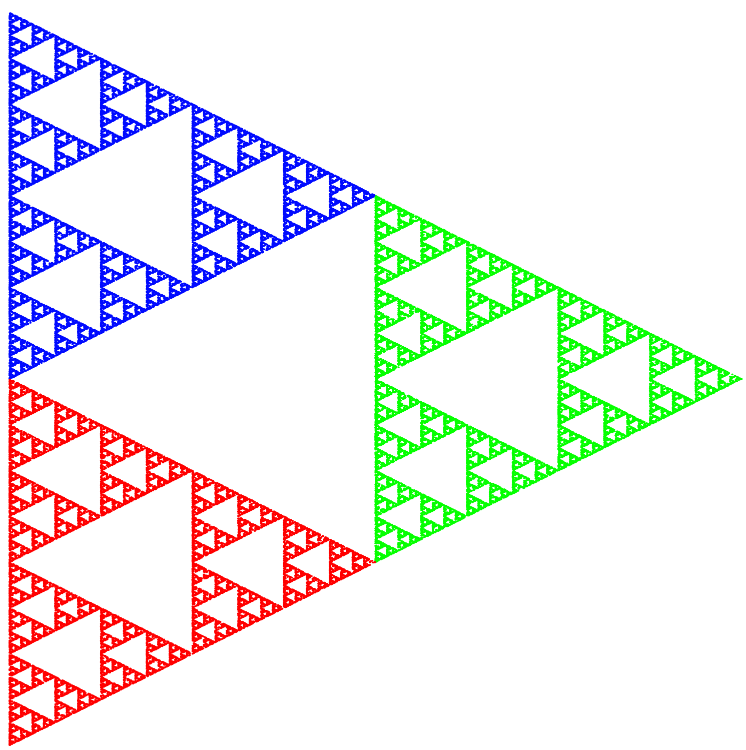


Figure 6. Subdiving the triangle and deleting the middle quarter

There are a number of ways of indicating the various points on this set. We will explore some of them and indicate methods of constructing.

2.3. The Tetrahedron. An equilateral tetrahedron can easily be embedded in 3-dimensional space. For example, you could consider the four points $(1, 0, 0)$, $(-1/2, \sqrt{3}/2, 0)$, $(-1/2, -\sqrt{3}/2, 0)$, and $(0, 0, \sqrt{2})$ to be the vertices of the tetrahedron. I leave it to you to check that each pair of points is at a distance of $\sqrt{3}$. A tetrahedron is often confused with a pyramid. The latter has four triangular faces, and one square face. The tetrahedron has four triangular faces, it has six edges, and four vertices. The tetrahedron of this paragraph has a slight disadvantage in that its vertices have irrational coordinates.

Consider, for a moment, the tetrahedron formed by the (x, y)-plane ($z = 0$ or floor), the (x, z)-plane ($y = 0$ or west), the (y, z)-plane ($x = 0$ or north), and the plane $x + y + z = 1$ in which all coordinates are non-negative. This tetrahedron is nicer than the previous one since its vertices are $(0, 0, 0)$, $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$. But it is not equilateral, and not all four faces are congruent.

Neither tetrahedron constructed is completely analogous to the previous constructions. To construct a segment, we looked in the plane, and to construct a triangle, we looked in 3-space. So now we are ready to venture out into 4-dimensional space to construct a tetrahedron.

If you ever read "A Wrinkle in Time," by L'Engle or "And He Built a Crooked House," by Heinlein, then you have been exposed to 4-dimensional space from the point of view of the novelist. In the arts, Picasso, and Duchamp were drawing using ideas from 4-dimensional geometry books. The idea of four spacial dimensions was popular up until the time of Einstein when he parameterized time as the 4-dimension. Current artists who explore the beauty of higher dimensional space include Tony Robbin.

Consider the set of four variables such that each is non-negative, and they sum to 1. The resulting set is the tetrahedron

$$S_3 = \{(x, y, z, w) \in \mathbb{R}^4 : x + y + z + w = 1, \quad \& \quad x, y, z, w \geq 0\}.$$

As you can see, we ran out of letters at the end of the alphabet, and so we had to back track a little. Later I will replace these letters by subscripted variables. We can start from the standard coordinate vectors $e_1 = (1, 0, 0, 0)$, $e_2 = (0, 1, 0, 0)$, $e_3 = (0, 0, 1, 0)$, and $e_4 = (0, 0, 0, 1)$. Then we take the intersection of all the convex sets that contain these 4-points. Before I describe the various facets of this sets let me say a bit more about 4-dimensional space.

The other day, I watched a weather report for a given city. The weather data consisted of several independent numbers: For example, temperature, barometric pressure, humidity, and ozone level for the day were displayed. If for each point in the city these four weather data were measured, then we could associate a point in 4-dimensional space to that point in the city. Had the city been on the prairie, then the weather coordinates together with the longitude and latitude would have defined a point in 6-dimensional space. If the city had tall buildings, we could have made weather measurements in yet another direction. My point is that many people consider time to be *the* fourth dimension, but it is only one of a number of other possible parameters that we can associate to describe a space around us. This paragraph returns to the question of what dimensional space do we live. The matter is irrelevant since functionally it is 3-dimensional. However, we should be philosophically ready to embrace higher dimensional reality even when we can only perceive it on a computational or algebraic level. One of the main purposes herein is to create point sets in higher dimensions by means of a computer program, rotate the images in higher dimensions, and project the results to the page of the paper. We can thereby get a glimpse of higher dimensional reality even though it is generated by computational devices.

Let us return to the tetrahedron. The four faces consists of the four triangles whose vertices are (e_1, e_2, e_3), (e_1, e_2, e_4), (e_1, e_3, e_4), and (e_2, e_3, e_4). Each such triangle is the intersection of all the convex sets that contain these vertices and we will call these F_4, F_3, F_2, and F_1, respectively. That is, the face is subscripted by the vertex that is opposite to it. The vertices can be paired, and each pair of vertices forms an edge. These segments have their end points (e_1, e_2), (e_1, e_3), (e_1, e_4), (e_2, e_3), (e_2, e_4), and (e_3, e_4). The edges are the intersections of all convex sets containing these pairs of points. As such each edge is identical to the edges of the first subsection; each has length $\sqrt{2}$. The faces each have area $\sqrt{3}/2$ since each face is an equilateral triangle that is identical to the one constructed in the second subsection.

We can even compute the volume of this tetrahedron as follows. The volume of any 3-dimensional conical set is given by $V = Ah/3$ where A is the area of the base and h is the height of the figure. The point $(1/3, 1/3, 1/3, 0)$ is on the face F_4. A line segment from $(0, 0, 0, 1)$ to this point is perpendicular to the triangle F_4. The length of this line segment is still given by the distance formula

$$\sqrt{(\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 + (\Delta w)^2} = \sqrt{1/9 + 1/9 + 1/9 + 1} = \frac{2\sqrt{3}}{3}.$$

Thus the volume is

$$V = \frac{\frac{\sqrt{3}}{2}\frac{2\sqrt{3}}{3}}{3} = \frac{1}{3}.$$

2.4. The n-simplex. Let n denote an arbitrary positive number. Consider the set of all $(n+1)$-tuples of non-negative numbers that add up to equal 1. That is consider the set

$$\begin{aligned} S_n &= \{(x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \\ & \quad x_1 + x_2 + \cdots + x_{n+1} = \sum_{j=1}^{n+1} x_j = 1, \\ & \quad \& \ x_j \geq 0 \ \text{ for all } \ j = 1, 2, \ldots, n+1\}. \end{aligned}$$

This set is what is called the *n-dimensional simplex.* By letting $n = 1, 2,$ or 3, we see that the segment is a 1-simplex, the triangle is a 2-simplex, and the tetrahedron is a 3-simplex. The next case would be a 4-simplex which I will describe in some detail in a moment. First, let me give some alternate descriptions.

The jth *standard coordinate vector*, e_j in $\mathbb{R}^{n+1}$ is defined to be the ordered $(n+1)$-tuple of numbers each of which is equal to 0 except for the number in the jth coordinate which is a 1. That is, $e_1 = (1, \underbrace{0, \ldots, 0}_{n})$, $e_2 = (0, 1, \underbrace{0, \ldots, 0}_{n-1})$, $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{n+1-j})$, $\ldots$, $e_{n+1} = (\underbrace{0, \ldots, 0}_{n}, 1)$.

We take the intersection of all the convex sets in $\mathbb{R}^{n+1}$ that contain these points. The intersection of all convex sets that contains a given set is called the *convex hull* of that set. The convex hull of a set is the smallest convex set that contains the given set. The convex hull of the set $B_{n+1} = \{e_1, e_2, \ldots, e_{n+1}\}$ is the n-simplex. Consider, for example, any pair of elements, e_j and e_k taken from the set B_{n+1} where $j < k$. Then for any number t between 0 and 1 inclusive (we can write $t \in [0,1]$ to mean $0 \le t \le 1$), the point

$$te_j + (1-t)e_k = (0, \ldots, 0, \underbrace{t}_{jth}, 0, \ldots, 0, \underbrace{1-t}_{kth}, 0, \ldots, 0)$$

is in the n-simplex, S_n. Similarly, if $x+y+z=1$ and each of x, y, and z is a non-negative number, then the triangle $xe_j + ye_k + ze_\ell$ is in the n-simplex S_n. We continue in this manner. Each of the tetrahedra, $xe_j + ye_k + ze_\ell + we_m$ where $x+y+z+w=1$ and $x, y, z, w \ge 0$ is in the n-simplex when $n \ge 4$. Let us count the number of each of these types of sub-simplicies.

For each pair j, k taken with $1 \le j < k \le n+1$, we have such a segment contained in the n-simplex. Thus the n-simplex contains $\binom{n+1}{2} = \frac{(n+1)n}{2}$ edges. For each triple j, k, ℓ with $1 \le j < k < \ell \le n+1$, we have such a triangle in the n-simplex. Thus the number of triangular facets is $\binom{n+1}{3} = \frac{(n+1)n(n-1)}{3 \cdot 2}$.

In general, the *binomial coefficient* $\binom{a}{b} = \frac{(a)!}{b!(a-b)!}$ is the number of b-element subsets of a set of size a. The exclamation point denotes the factorial function $a! = a(a-1)!$ where by definition $0! = 1$. In the n-simplex, there are facets that are q-simplices; the number of these lower dimensional facets is exactly the binomial coefficient $\binom{n+1}{q+1}$.

Binomial coefficients are useful if you want to compute the chances of winning a lottery. Suppose that a given state has a six number lottery in which the numbers are chosen out of the set of numbers from 1 to 49. Then the number of possible combinations is $\binom{49}{6} = 13,983,816$. Most state lotteries start their initial payoff to be one-half of the number of possible combinations, and let the pot grow if a winning ticket is not bought. So the initial pay-off is usually around \$7,000,000. When the pot gets above $14,000,000$, then more people buy tickets because they feel the expected worth of the ticket (the payoff times the probability of winning) is less than the cost. A problem with that strategy is that when more people buy tickets, the probability that the pot will be split goes up.

For the record, I occasionally buy lottery tickets, but have not yet won. An alternative way of making extra money is to write books, and to hope someone else likes them. If I won the lottery, I would have more time to write books, and I could make more money. The rich get richer ….

Consider the 4-simplex, S_4, which is the convex hull of the set $B_5 = \{e_1, e_2, e_3, e_4, e_5\}$. For convenience and because I will use this idea later, I will represent the five standard coordinate vectors in 5-space by five equally spaced points on a circle of radius 1. These points are subtended by angles of $0°$, $72°$, $144°$, $216°$, and $288°$. These angles are all multiples of $72° = 360°/5$. Associate e_1 to the point at $72°$ and then proceed in order. So e_2 goes to $144°$, and e_5 goes to $360°$ which is also the point subtended by a $0°$ angle. Since everything is a multiple of 72°, we can just as easily call these points 1 through 5. Now between any two of these points draw a line. The process is illustrated step-by-step in Fig. 7. At the end of the process, we obtain the familiar pentagram. Each triple of points represents a triangle, and each quadruple represents a tetrahedral face. So the edges of 4-simplex projects into the plane as the pentagram, and two tetrahedral faces that intersect along a triangle have three vertices in common.

The pentagram is a symbol that is currently associated with witchcraft and Satanism. In the tale of "Sir Gawain and the Green Knight," it was thought to be a Christian symbol. That it expresses the configuration that is of a fundamental nature to 4-dimensional space may explain the mysticism that is associated. Renaissance artists, such as Leonardo Da Vinci, consciously used the symbol and its ancillary ratios conspicuously in their sacred art. The ratios $\phi = (\sqrt{5} \pm 1)/2$ are related to the Fibonacci sequence as limits of the ratios of these numbers. This ratio (albeit an irrational number) is called the divine ratio. The renaissance thinkers obviously thought that rational thought and mathematical ideas were in the province of the deity. By studying mathematics, we are learning to understand the inner-workings of the human mind. If human kind is created in the image of the deity, then by studying math, we are able to glimpse at the deity's thoughts. In the meantime, one should not ascribe too much meaning to symbols, names, and notation. Sometimes they are chosen merely for convenience.

For an n-simplex, we can associate the standard coordinate vectors to the $(n+1)$ evenly spaced points on the unit circle that are subtended by angles whose degree measurement is a multiple of $360°/(n+1)$. The edges are the set of lines that join pairs of these points. They can be labeled as follows:

$$\begin{array}{ccccc}
(1,2) & (1,3) & & \ldots & (1,n+1) \\
 & (2,3) & & \ldots & (2,n+1) \\
 & & & \vdots & \vdots \\
 & & & (n-2,n+1) & (n-1,n+1) \\
 & & & & (n,n+1)
\end{array}$$

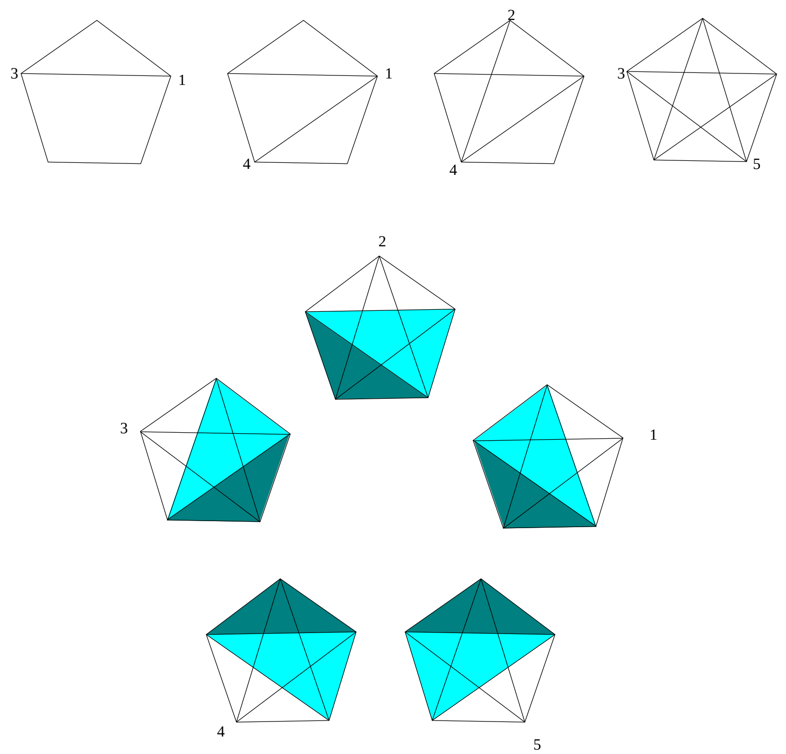


Figure 7. The edges in a 4-simplex and five tetrahedral facets

Similarly, the names of the triangular faces of the n-simplex can be arranged in an array that is shaped like a tetrahedron, and the names of the q-dimensional faces can be arranged in the shape of a $(q+1)$-simplex. I urge you to attempt these enumerations for simplices of dimensions 7 and less.

Figure 8 indicates the edges in an 8-simplex. These form a nice pattern on a nonagon. The ring that Albert's mother wears was also worn by my mother. It is a nine cut diamond. It was not unusual for me to draw these patterns with pencil, compass, and protractor when I was young. I gained a great deal of insight and intuition from these drawing endeavors. If you have children, you might encourage them to draw such figures, or you might try to draw them yourself.

If you do so, and draw the edges systematically, you will certainly understand why the edges of the n-simplex are naturally arranged in a triangular array.

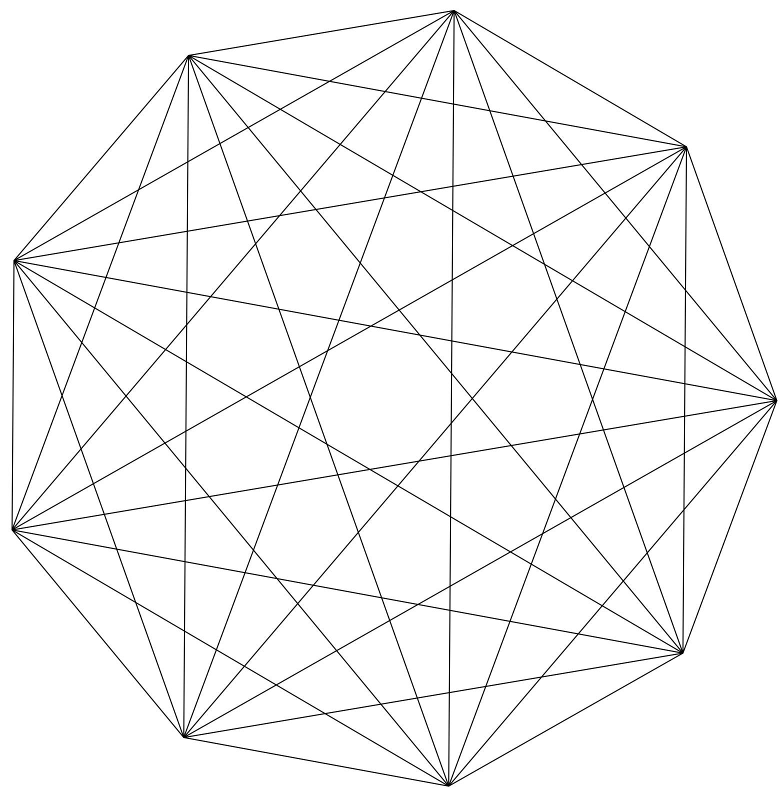


Figure 8. The edges of the 8-simplex

It is interesting to push analogies as far as they can go. During our discussion of the edge, triangle, and tetrahedron, we computed the length, area, and volume of each. The sequence of these numbers is

$$\frac{\sqrt{2}}{1}, \frac{\sqrt{3}}{2}, \frac{\sqrt{4}}{6}.$$

We can use mathematical induction to compute the hyper-volume of the n-simplex as follows. The n-simplex is created from the $(n-1)$-simplex by a "cone" construction. A point not in the hyperplane that contains the $(n-1)$-simplex is joined to each point of the $(n-1)$simplex to form an n-simplex. The point $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ in which n-coordinates are mentioned is clearly a point on the $(n-1)$-simplex. The "cone-point" of the n-simplex is the point e_{n+1}. The distance from this point to the central point, $(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}, 0)$, is given by the radical

$$\sqrt{n \cdot \frac{1}{n^2} + 1} = \sqrt{\frac{n+1}{n}}.$$

Now in $(n+1)$-space, the hyper-volume of a conical figure is given by the formula

$$\mathcal{V}_{n+1} = \frac{h\mathcal{V}_n}{n+1}.$$

(The formula can be derived from calculus using multiple integrals. Alternatively, it can be demonstrated by means of an analysis of similar triangles that arise in the cone construction, estimating hyper-volumes using hypercubes, and summing powers of the integers.) So assuming that the volume of an $(n-1)$-simplex is

$$\mathcal{V}_n = \frac{\sqrt{n}}{(n-1)!},$$

we obtain

$$\mathcal{V}_{n+1} = \frac{h\mathcal{V}_n}{n+1} = \frac{\sqrt{\frac{n+1}{n}}\frac{\sqrt{n}}{(n-1)!}}{n} = \frac{\sqrt{n+1}}{n!}.$$

That is to say that the formula works for the lower dimensional cases, and if it works for the case $n-1$, then it works for the case n.

The $n!$ in the denominator indicates that the hyper-volume of the n-simplex tends to be less of a percentage of the volume of a hyper-cube as n gets large. In the hyper-cube, dust-bunnies in the corner are more insignificant than in 3-space.

To complete the analogy with the lower dimensional cases, I point out that the arrow in $(n+1)$-space that points from $(1/(n+$

$1), \ldots, 1/(n+1))$ to $((n+2)/(n+1), \ldots, (n+2)/(n+1))$ is perpendicular to any point on the n-simplex. In fact, this follows from the formula for the n-simplex as the solution set of the equation

$$x_1 + x_2 + \cdots + x_{n+1} = 1$$

where $x_j \geq 0$ for $j = 1, 2, \ldots, n+1$. The n-simplex is the higher dimensional skim board. The vector perpendicular to it is the line from the center of gravity of the hyper-rider to the board. The board stays on plane by means of the hyper-rider's weight.

3. The Chaos Game

This section describes how I constructed the sets of points in the n-simplex that you are viewing in the flip-books in the margins. First a comment on the "game." On his web site,

`http://math.bu.edu/DYSYS/applets/chaos-game.html`

Bob Devaney describes a game whose object is to hit a particular triangle in a step in the Sierpinski triangle by starting at a particular vertex and moving half-way toward one of the other vertices in the triangle. On each move, the player moves halfway towards another vertex. The object is to land within the given triangle in the smallest number of moves. After one has played a few times, one can determine a strategy to hit the triangle by establishing an address for the triangle. The addressing process will be discussed below.

3.1. Chaos in the Triangle. Suppose that an infinite sequence of numbers is given in which each number in the sequence is either $1, 2$, or 3. There are uncountably many such sequences. Some of the boring ones are $(1, 1, 1, \ldots)$, $(2, 2, 2, \ldots)$, $(1, 2, 1, 2, \ldots)$, and so forth. They are boring since they are *periodic*; once an initial pattern is established, then it repeats forever. We are interested in the not so boring sequences; these are the non-repeating, non-periodic sequences. The not boring sequences may be achieved by a pattern, such as $(1, 2, 2, 3, 3, 3, 1, 1, 2, 2, 2, 3, 3, 3, 3, \ldots)$, or they may be completely random. In order to generate such a sequence, I use a pseudo-random number generator that my computer program creates. Of course, I do not establish an *infinite* sequence via the computer program, but I create a very long segment. The figures generated on the disk used sequences between 70,000 and 100,000 long. The reason for stopping after 70,000 was that the computer could not easily handle more than these. Moreover, pixels and dots on the paper have diameter, so if too many points are chosen, then the pictures get blurry. The figures in the margins use far fewer points; the figures are small and only a few points are needed to express the ideas.

In principal, the constructions that I describe can work with infinite sequences even though we cannot actually implement them with an infinite number of points in practice. The points in the infinite sequence are named $A = (a_1, a_2, a_3, \ldots)$.

Plot the point, e_{a_1}, where a_1 is the first number in the infinite sequence. Let this point be labeled b_1. The next step is to look at the next number, a_2, in the sequence, and then move half-way towards the corresponding vertex, e_{a_2} and plotting this midpoint, b_2, in the triangle. Having plotted the kth point, b_k, in the triangle, plot the $(k+1)$st point by: (1) examining the $(k+1)$st number, a_k, in the infinite sequence (2) measuring the distance from the kth plotted point to the vertex e_{a_k}, and (3) letting b_{k+1} be the point that is the midpoint between b_k and the vertex $e_{a_{k+1}}$. The process is illustrated for a few steps in Fig. 9.

Two comments are needed before we proceed. The first is that we could move closer than halfway to the next point. In the fourth marginal flip books the distance to the next point was taken to be 3/8. The second comment is that the game should not be limited to a triangle. It could be played in any convex figure whether it be planar, 3-dimensional, or higher dimensional. Convexity is needed to make sure that the points plotted are within the set: If every point on the line joining two points is within the set, then, in particular, the midpoint is in the set.

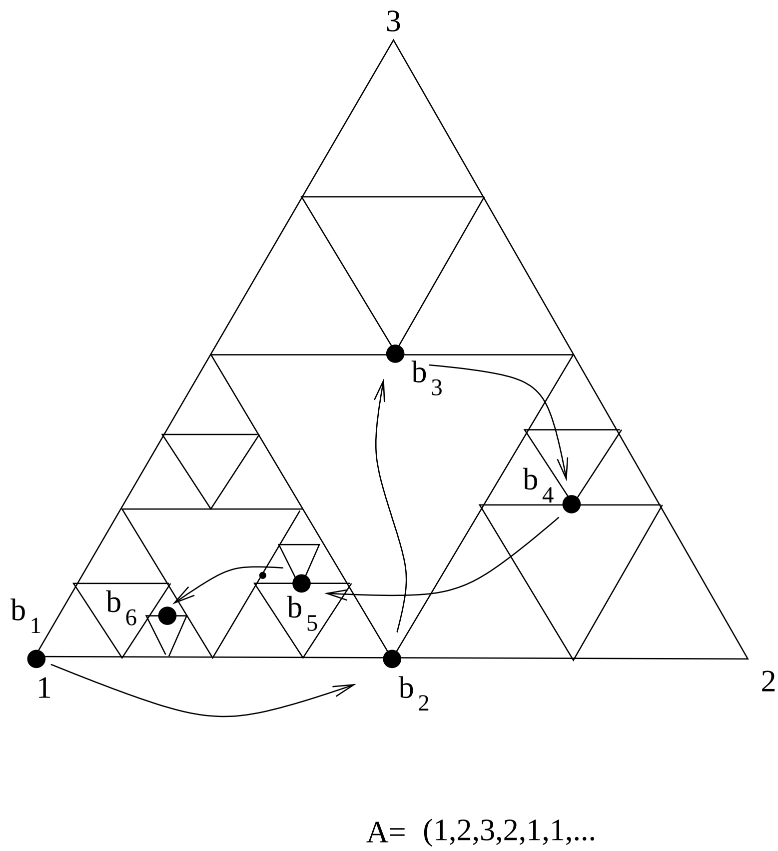


Figure 9. The first few moves in the chaos game

The chaos game is found among many mathematical popularizations. Albert worked two science fair projects in regard to the chaos game. In the first, Albert plotted figures in regular polygons by using the sequences of digits in the n-ary expansion of certain irrational numbers. An *n-ary* expansion is like a decimal expansion except that instead of the "decimal" part being a power of ten, it is a power of some number like 3, 4, 5 etc. The digits in an n-ary expansion are found among the set $\{0, 1, \ldots, n-1\}$. For example, the fraction $1/3$ in ternary expansion is $(0.1000\ldots)_3$. In binary expansion, $1/3 = 1/4 + 1/16 + 1/64 + \cdots$, so $1/3 = (0.01010101\ldots)_2$. An irrational number will have non-repeating n-ary expansion for any $n > 1$. The second project involved following the chaos game in the five platonic solids (cube, tetrahedron, octahedron, icosahedron, and dodecahedron). While viewing the chaos game in the tetrahedron, he saw a projection in which the figure resembled the chaos game in the square. This lead us to think that the projection of the chaos game in the n-simplex would resemble the chaos game in a regular polygon with $(n+1)$ vertices. Indeed, it is and we will demonstrate this fact after we discuss the projections.

3.2. Chaos in the n-simplex. Rather than recapitulating the chaos game in the triangle with more notation, we turn now to the general description. Let $A = (a_1, a_2, a_3, \ldots,)$ denote an infinite sequence of numbers where $a_j \in \{1, 2, \ldots, n+1\}$. Let

$$b_1 = e_{a_1},$$

and let

$$b_k = (1/2)b_{k-1} + (1/2)e_{a_k}.$$

That is the kth point in the n-simplex is chosen to be the midpoint of the line segment that connects the $(k-1)$st point and the vertex, e_{a_k}, whose subscript is the kth number in the sequence A. The set

$$B_A = \{b_k \in \mathbb{R}^{n+1} : k = 1, 2, \ldots \quad b_1 = e_{a_1} \quad b_k = (1/2)b_{k-1} + (1/2)e_{a_k}\}$$

is the *image of the chaos game* in the $(n+1)$-simplex as it depends on the initial sequence A.

Let us follow this idea for a ternary (base 3) sequence for the first few steps. Suppose that a given sequence, A begins

$$(1, 2, 3, 2, 1, 1, 2, 3, \ldots)$$

$$= (a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8, \ldots).$$

Then we plot the point $b_1 = e_{a_1} = e_1 = (1, 0, 0)$. To find b_2 we add $(1/2)b_1 + 1/2e_{a_2} = 1/2(1, 0, 0) + 1/2e_2 = (1/2, 0, 0) + (0, 1/2, 0)$ $= (1/2, 1/2, 0) = b_2$. The value $b_3 = (1/2)b_2 + 1/2e_3 = (1/4, 1/4, 1/2)$ Furthermore, $b_4 = (1/8, 1/8, 1/4) + 1/2(0, 1, 0) = (1/8, 5/8, 1/4)$. You may wish to compute the next few values, or not, if the pattern is clear to you.

If the sequence A is boring (periodic), then the image B_A will consist of finitely many points in the n-simplex. If two sequences A and A' are sufficiently general (for example each is generated by a random process), then the image B_A and $B_{A'}$ will resemble each other to a very high degree of accuracy. We will show this by demonstrating that both B_A and $B_{A'}$ have all their points land in a certain set called the Sierpinski n-simplex. Moreover for any point in the Sierpinski n-simplex and for any arbitrarily small distance the probability that a point in B_A lands within that distance of the given point is as close to 1 as you would like.

3.3. The Sierpinski n-simplex. I do not know if Sierpinski thought about the general construction that I give here. The Sierpinski triangle is a well known object, and Sierpinski was a smart dude, so I expect that he would recognize this construction in general.

Before I describe the process, I need to define a few terms. An *open neighborhood of radius* $\epsilon > 0$ *of a point* $\vec{x} \in \mathbb{R}^{n+1}$ consists of those points $\vec{y} \in \mathbb{R}^{n+1}$ whose distance from $\vec{x}$ is strictly less than ϵ. This neighborhood is denoted $N_\epsilon(\vec{x})$. If the coordinates of these points are given as $\vec{x} = (x_1, x_2, \ldots, x_{n+1})$ and $\vec{y} = (y_1, y_2, \ldots, y_{n+1})$, then the distance between them is

$$d(\vec{x}, \vec{y}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_{n+1} - y_{n+1})^2)} < \epsilon.$$

The *interior of a set* Y is the union of all the ϵ neighborhoods of points in Y for which the neighborhood is contained entirely within Y. The interior of the ocean is all the water below the surface. The concept of a set's interior is a fundamental concept in mathematics, but to discuss it in detail here, now, would take us far afield. The idea is that the interior is everything on the inside. In the next paragraph, we will remove the interior of sets so that the boundaries remain.

The midpoints of the lines joining the standard coordinate vectors e_j in $\mathbb{R}^{n+1}$ are $(n+1)$-tuples of numbers which have exactly two non-zero coordinates and these are both 1/2. There are $\binom{n+1}{2} = \frac{(n+1)n}{2}$ such midpoints within the n-simplex. We consider the convex hulls of these midpoints and remove from the n-simplex the interior of this convex hull. (Recall, that the convex hull is the intersection of all the convex sets that contain a given set). The resulting set will have $(n+1)$-pieces. These pieces are connected to each other at some of their vertices. To be specific, we will consider the piece of the n-simplex that contains the vertex e_1. Nearby, there are the vertices $(1/2, 1/2, 0, \ldots, 0)$, $(1/2, 0, 1/2, 0, \ldots, 0)$, $\ldots (1/2, 0, \ldots, 1/2)$. We have a mini-copy of the n-simplex in the corner that contains e_1 because each of these vertices are at a distance $1/\sqrt{2}$ from e_1 and each is in the direction e_j. So each of the resulting pieces of the n-simplex is a mini-simplex.

In an n-simplex, regardless of its size, every vertex is connected to every other vertex by an edge in the simplex. Each of the mini-simplices looks identical to the original simplex, but is of a smaller scale. So we can repeat the process of removing the interior of the convex hull of the midpoints of each edge in the mini-simplices. We get more mini-simplices, and continue the process in each of these.

Every pair of mini-simplices intersect at the half-point vertices. Nevertheless, we can give an address to every point that remains after the removal of all of the interiors. The description goes as follows. Label the mini-simplex that contains the point e_j with the number j. After the middle interior of it has been removed, then the result resembles the big simplex with its middle interior removed. The region that contains e_j will be labeled (jj) that which contains $1/2e_j + 1/2e_k$ will be labeled (jk). Notice that the (jj) region also satisfies the condition that it contains this point, but there is a unique other such region. Out of each of these simplices, the middle interior is also removed. The region that contains $1/2e_j + 1/4e_k + 1/4e_\ell$ will be labeled $(jk\ell)$.

We continue by induction. Label the region that contains the point $\frac{1}{2}e_{j_1} + \frac{1}{4}e_{j_2} + \cdots + \frac{1}{2^k}e_{j_k}$ be labeled by the finite sequence $(j_1, j_2, \ldots, j_k)$. These labelings are shown in Figs. 11, 12, and 13. In the triangle, the method for obtaining the remaining labels should be apparent. In the tetrahedron and the 4-simplex, the labels are more difficult to read since the figures are drawing of projections of the figures (See Section 5 for a description of how the 4-simplex is projected to the plane).

Since we continue removing the interiors of all the convex hulls of all the midpoints of all the simplices and mini-simplices and mini-mini-simplices and so forth on to infinity, the addresses of the points in the Sierpinski sets are given by infinite sequences $(j_1, j_2, j_3, \ldots)$ with each $j_k \in \{1, 2, \ldots, n+1\}$. Furthermore, the Sierpinski sets have no interior left.

A particularly delicious drink that is served in the Varsity Drive-In (the world's largest drive-in located in Atlanta and Athens, Ga) is called a frosted orange. It is created in a milkshake machine from an orange flavored milk drink. Such drinks have an emulsifier that is based upon sea foam which is in turn derived from seaweed. The existence of a seaweed emulsifier in a frosted orange is purely conjecture on my part, but the rest of this story bears out this possibility. Namely, one night a drummer friend of mine ordered a frosted orange to-go, didn't drink it, but left it behind his drum set on a late night gig. The next afternoon, when he came to pick up his trap set, the frosted orange had dried out, but retained its ice cream shape. This matrix was, in my opinion, a manifestation of the seaweed emulsifier in the drink. The Sierpinski sets are like this glutinous matrix. It is what remains when all the possible interiors have been removed. Yet it retains its character and indeed describes the shape from whence the interiors were removed. Robert, the drummer, doesn't drink frosted oranges any more.

The *Sierpinski n-simplex*, then, is the end result of an infinite process: In the *kth* stage of the process, the interior of the convex hull of of the central portion of each mini-simplex is removed. The number of mini-simplicies that remain in the previous ($(k-1)$st) stage is $(n+1)^{k-1}$ and so the number of mini-simplices in the kth stage is $(n+1)^k$. When k is big $(n+1)^k$ is humongous. The process goes on forever.

The central piece that is removed has an interesting shape. In the triangle, it is an upside-down triangle. In the tetrahedron it is an octahedron. In the n-simplex, some of its facets are identical to the part removed from the $(n-1)$-simplex. Others are $(n-1)$dimensional simplices. For example, in the 4-simplex we remove the interior of the convex hull of the ten points

$$\begin{array}{rrrr}
(1/2,1/2,0,0,0) & (1/2,0,1/2,0,0) & (1/2,0,0,1/2,0) & (1/2,0,0,0,1/2) \\
 & (0,1/2,1/2,0,0) & (0,1/2,0,1/2,0) & (0,1/2,0,0,1/2) \\
 & & (0,0,1/2,1/2,0) & (0,0,1/2,0,1/2) \\
 & & & (0,0,0,1/2,1/2)
\end{array}$$

In the five tetrahedral faces of the 4-simplex, there are five octahedra that are removed. For example, consider the six vertices (among those in the list above) whose last coordinate is 0. The convex hull of these is an octahedron. The remaining 4-vertices form a tetrahedron. The set of lines that joins these two figures forms the boundary of the shape that is removed. The figure removed can be similarly decomposed around any of these ten points in which one coordinate is non-zero (there are six such points), and the other four in which the given coordinate is 1/2. Thus there are five octahedra and five tetrahedra in the boundary of the figure. Figure 10 makes an attempt to illustrate this shape; we hope that in the flip book you will look for it.

Once the first removal is accomplished, the subsequent removals can be achieved by scaling all the lengths by 1/2, and performing similar removals in each of the resulting $(n+1)$-daughter simplices. Consequently, exponentially more hulls are removed at each stage.

A point in the Sierpinski n-simplex can be described by an infinite sequence of numbers taken from $\{1, 2, \ldots, n+1\}$. There is another method for describing the points in the Sierpinski n-simplex (given in a moment), but the addressing process is very convenient in terms of locating points from the chaos game. The reason for the convenience goes as follows. In the chaos game the first point is going to be one of the vertices of the original simplex. The second point plotted will be the corresponding vertex of the simplex that is closest to the second point in the infinite sequence A. So if A starts $A = (1, 2, \ldots)$, then the first point of the mini-simplex is the second point plotted. The

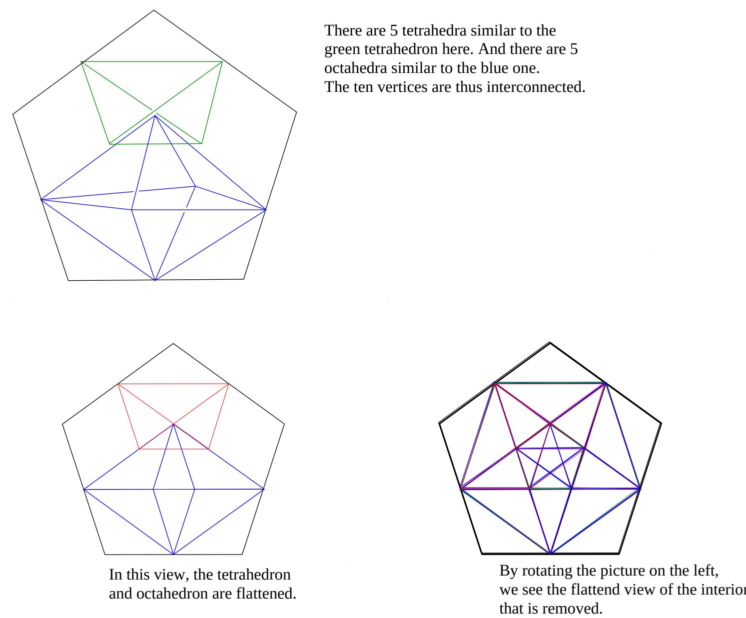
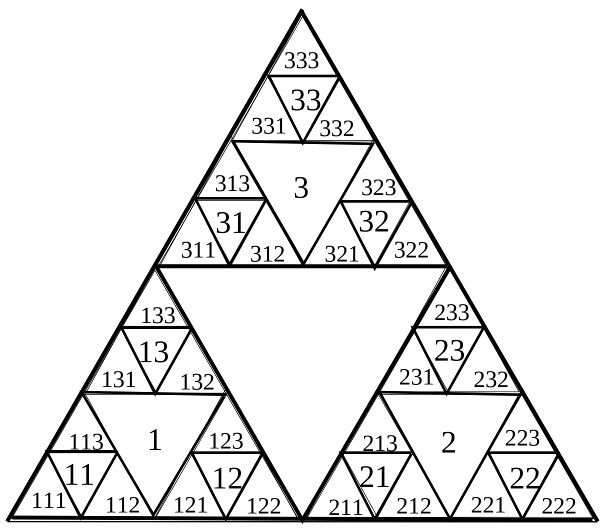


Figure 10. The middle piece that is cut from the 4-simplex

Figure 9 illustrates what happens, and the general pattern can be gleaned by comparison.

Specifically, in the figure we see the left vertex of the triangle is the first point in the chaos game. This point lands in the triangle that is labeled 1. The second point in the chaos game lands at the left point of the second triangle or the triangle that is labeled (21). The third point lands at the left point of the upper right triangle—the triangle labeled (321). In general, the kth point in the chaos game lands in the triangle whose address is $(a_k, a_{k-1}, \ldots, a_1)$ or the triangle whose address is a backwards expression of the original sequence A. The same phenomena happens in higher dimensions, but instead of triangles, the localities are simplices.

I could have easily addressed the sub-simplicies with the reversed address, and I would have rewritten this section with those addresses, but I would have had to also redraw the figures.



The first three steps in the labeling scheme for the triangle

Figure 11. Labeling the regions in Seipinski's triangle

The coordinates of the points in the Sierpinski n-simplex are found among the set

$$D_n = \{(x_1, x_2, \ldots, x_{n+1}) : \sum_{j=1}^{n+1} x_j = 1 \ \& \ x_j = \frac{a}{2^k} \text{ for all } j = 1, \ldots, n+1\}.$$

This is the set of *dyadic rational numbers* — rational numbers whose denominators are powers of 2. Observe that the coordinates of the corners of the mini-n-simplices all must be dyadic rationals by construction: The number 1 is a dyadic rational because its denominator can be chosen to be $2^0 = 1$. After the kth stage, the denominators range from 2^0 to 2^{k+1}. Then in the $(k+1)$st stage points are introduced at the corners whose coordinates are half of those in the

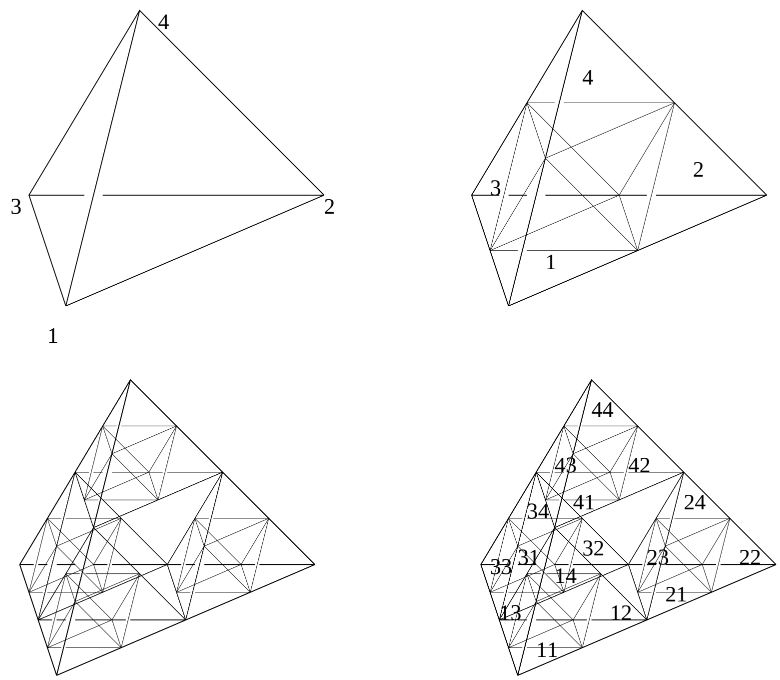


Figure 12. Labeling the regions in the tetrahedron

previous stage. This takes care of the corners but what about the interiors? There is no interior! We have removed all the interiors of all the mini-simplices. If any open set remained, it would contain the interior of a mini-simplex, but they have all been removed.

Also observe that the chaos game gives points in the Sierpinski n-simplex. The first point in the chaos game is $b_1 = e_{a_1}$ where the initial infinite set $A = (a_1, a_2, \ldots)$. The next point is $b_2 = 1/2e_{a_1} + 1/2e_{a_2}$; the coordinates are both 1/2 clearly dyadic rational numbers. If the coordinates of b_k are dyadic rationals, then the coordinates of b_{k+1} are $1/2b_k + 1/2e_{a_{k+1}}$ which by induction is a dyadic rational.

If you and I were to have a logical discussion in which we both learned about the dyadic rational numbers we could call this a didactic dyadic dialectic.

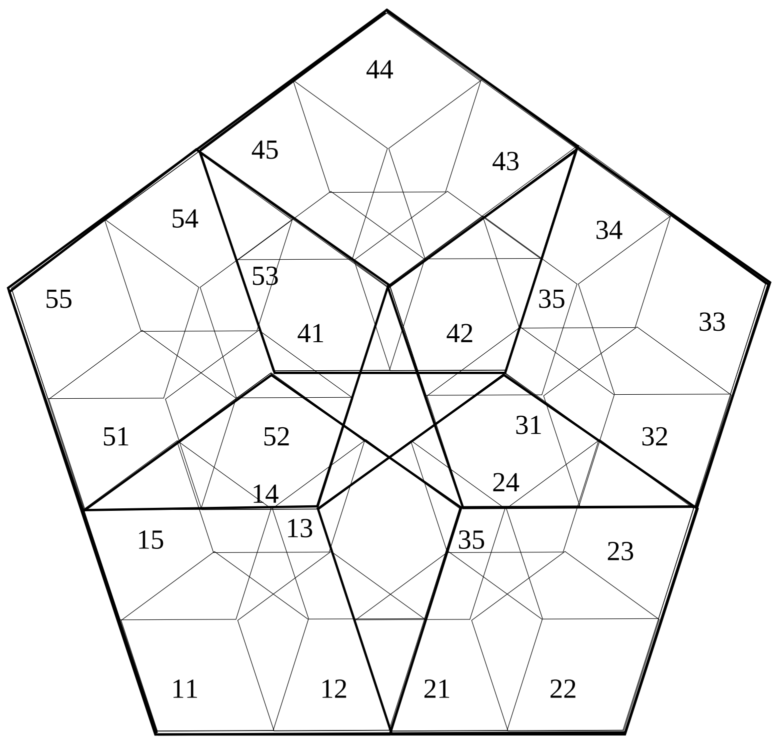


Figure 13. Labeling the 4-simplex

Beauty is often found in paradoxes. Two facts about the Sierpinski n-simplex, when taken together, are seemingly paradoxical. First, the chaos game cannot possibly cover all of the points in the Sierpinski n-simplex. Second, for any point in the Sierpinski n-simplex, and for any arbitrarily small distance, there is a point from the chaos game that lies within that distance from the Sierpinski n-simplex. These facts are in my opinion so bizarre that the both require proof. So we will restate them in the theorem/proof style. But we need to have a digression about the different types of infinity.

3.4. Infinity: Countable and Uncountable. *Finite sets* are sets that can be put into one-to-correspondence with the set $\{1, 2, \ldots, n\}$ for some $n = 1, 2, \ldots$. This means that the elements of the set can be labeled $\{y_1, y_2, \ldots, y_n\}$. A finite set cannot be put into one-to-one correspondence with any proper subset. The *natural numbers* $\mathbb{N} = \{1, 2, 3, \ldots\}$ can be put into one-to-one correspondence with the positive even numbers (natural numbers that are divisible by 2), every even number is natural, so the natural numbers are not finite; *i.e. infinite*. On the other hand the natural numbers can be *listed*: they can be arranged in an infinite list (as we have done), where every number has a successor. A set that is either finite or that can be listed is called *countable*. The natural numbers are *countably infinite*. This concept is sort of hard to get, but to count something is to put it into one-to-one correspondence with a subset of the natural numbers: There is a one-to-one [$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$] onto [For every $y \in Y$, there is an $n \in \mathbb{N}$ such that f(n)=y] function $f : \mathbb{N} \rightarrow Y$. So every element of the set Y has a uniquely associated number. The hard part to get is that there are infinite sets that are uncountable.

Among the first known sets that were found to be uncountable were sets that were discovered by Cantor, and the Sierpinski n-simplices are akin to Cantor's sets. Cantor displayed a way-cool proof that these sets are uncountable, and I will show you the proof in a moment. Before I do, I need to say that uncountable infinite is really big, bigger than most things that you have imagined in the past. Also, I should say that in Cantor's time his ideas were thought to be somewhat un-orthodox, and after his made these discoveries, he went crazy. So there should be some cautionary statements made.

It is popular to think, and media re-enforce this prejudice, that mathematicians are a bunch of crazy people. Movies such as "A Beautiful Mind"makeindexA Beautiful Mind and "Good Will Hunting" might suggest to the public that the field of mathematics is permeated with people who suffer from mental illness. In fact, some of my colleagues suffer from a variety of maladies: Ausberger's, Turrett's, bipolarism, obsessive compulsive disorders, and yes, there are paranoid schizophrenics among us. But the overwhelming majority of the mathematicians that I know are ordinary people with ordinary lives, and an extraordinary capacity for abstraction. I suspect that the statistics will indicate that the number of people with mental disorders among the mathematically talented population matches that of the general population. Math will not make you go crazy, and you don't have to be crazy to learn to do math.

Cantor's ideas were un-orthodox, but they were correct. They held up to the scrutiny of logical proof. Proof in mathematics is the essence of certainty. If you and I agree with a set of axioms, and these are consistent, then conclusions that can be drawn from these axioms are also agreed upon. Mathematicians are, for the most part, in the business of studying what properties follow from the axioms. Thus we construct mathematical entities (sets of numbers, geometric objects, equations, or their solutions) and study their properties. Often, the game that is involved to to study similarities or differences between entities. What is even more cool is that the collection of similarities can form entities which in turn can be discussed. Cantor's ideas, and Russell's subsequent codification of mathematics put some limits on the linguistics of studying similarities of similarities of ... of entities. So that although we can indeed study and understand infinite sets, we cannot go on forever thinking of sets of sets.

Here is Cantor's idea:

Theorem 3.1. *The Sierpinski n-simplex is an uncountably infinite set.*

Proof. The points in the Sierpinski n-simplex can be determined by their addresses as the collection of the infinite sequences $(c_1, c_2, \ldots,)$ where each $c_j \in \{1, 2, \ldots, n+1\}$. Suppose that we could list the set of all these infinite sequences and arrange them in a rectangular array:

$$\begin{array}{ccccc} c_{11}, & c_{12} & c_{13} & c_{14} & \cdots \\ c_{21}, & c_{22} & c_{23} & c_{24} & \cdots \\ c_{31}, & c_{32} & c_{33} & c_{34} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{array}$$

Each row in the array is an address in the Sierpinski set. Now we follow Cantor's idea and construct a point that is not in this infinite list. Namely, for each $j = 1, 2, 3, \ldots$, we let $d_j \in \{1, 2, \ldots, n+1\}$ be any number not equal to c_{jj}. That is we read along the diagonal and construct an address which differs from all the addresses that have been listed. If the original list held all the addresses it would have held the address $D = (d_1, d_2, d_3, \ldots)$. But since the address D disagrees with all the addresses in the list, then there is an address that the list missed. You can't exactly tack D onto the end of the list; there is no end to the list. Thus Sierpinski's n-simplex contains uncountably many points. □

On the other had the sequence of points obtained from the chaos game is countable. Starting with the sequence $A = (a_1, a_2, a_3, \ldots)$ we plot the points

$$b_1 = e_{a_1}$$
$$b_2 = 1/2e_{a_1} + 1/2e_{a_2}$$
$$b_3 = 1/2(1/2e_{a_1} + 1/2e_{a_2}) + e_{a_3}$$
$$\ldots$$
$$b_n = 1/2b_{n-1} + 1/2e_{a_n}$$
$$\ldots$$

By construction, this list is countable albeit infinite.

3.5. How Close Can You Get? Consider for a moment the real numbers. Remember, these are the set of points on a line, the set of infinitely precise measurements, the set of numbers that can be expressed by infinite decimals. There is a great similarity between the real numbers and the set of points in the Sierpinski n-simplex. In fact, any real number has an $(n+1)$-ary expansion, and so any point in the Seirinski n-simplex corresponds to a real number. However, two decimals can represent the same real number. The favorite example (in decimal notation) is the real number $x = 0.999\ldots$. This decimal represents the number $1 = 1.0000\ldots$. This fact is startling so we will explicate.

A friend of mine has a favorite proof that $x = 1$. If $x \neq 1$, then there must be a number between x and 1. Name such a number … You can't, right? Because there isn't one. The more traditional proof goes as follows: $10x - x = 9$: multiplying by 10 moves the decimal point once to the right. Now solve this equation for x. Alternatively, recall that $1/3 = 0.3333\ldots$ and $2/3 = 0.6666\ldots$, then $1/3 + 2/3 = 0.9999\ldots$.

The other fact, that I want you to remember, is that given any real number $y \in R$, and given any arbitrarily small distance $\epsilon > 0$, there is a rational number that is within a distance ϵ from y. The proof of this fact is remarkably easy. Choose m such that 10^{-m} is less than ϵ. You should figure out how this can be done. Then truncate the decimal expansion of y in the $3m$th digit. The factor of 3 is overkill, but hey, you want to be sure don't you? The truncated decimal is a rational number.

A fact that you may not know is that the rational numbers, $\mathbb{Q}$, form a countable set. We don't need this fact, but it too is easy. Think of a rational number as a ratio (a/b) where a, b are natural numbers. Associate to this ratio, the point in the plane (a, b). For convenience just deal with positive numbers; the general case will follow easily. Now it is possible to walk around the positive portion of the plane, starting from $(0, 1)$, and zig-zagging along covering every point whose coordinates are natural numbers. Such a path gives a function that is onto the rational numbers. Points such as $(1, 2)$, $(2, 4)$,$(3, 6)$, all represent the same rational number; that's ok, just don't count the redundancies. Figure 14 illustrates how to form the zig-zag path.

Let us recapitulate. The real numbers and the points in the Sierpinski n-simplex are both uncountably infinite sets. They can both be approximated to an arbitrary degree of accuracy by a countably infinite set. The paragraphs above indicate the process for approximating the reals by the rationals, and why the rationals are countable. We have shown the Sierpinski n-simplexes have uncountably many points. What remains to be shown is that the chaos game can approximate any point in the Sierpinski n-simplex with an arbitrarily high degree of accuracy.

Theorem 3.2. *Let $\epsilon > 0$, and let a point $\vec{y} \in \mathbb{R}^{n+1}$ be given to be a point in the Sierpinski n-simplex. Let $A = (a_1, a_2, a_3, \ldots)$ be a random sequence of numbers with $a_j \in \{1, 2, \ldots, n+1\}$. And let $b_j = \frac{1}{2}b_{j-1} + \frac{1}{2}e_{a_j}$ (with $b_1 = e_{a_1}$) be the points in the image of the chaos game that are determined by the sequence A. Let $B_A = \{b_n\}$. Then there is a a point $\vec{b} \in B_A$ such that the distance from $\vec{b}$ to $\vec{y}$ is less than ϵ.*

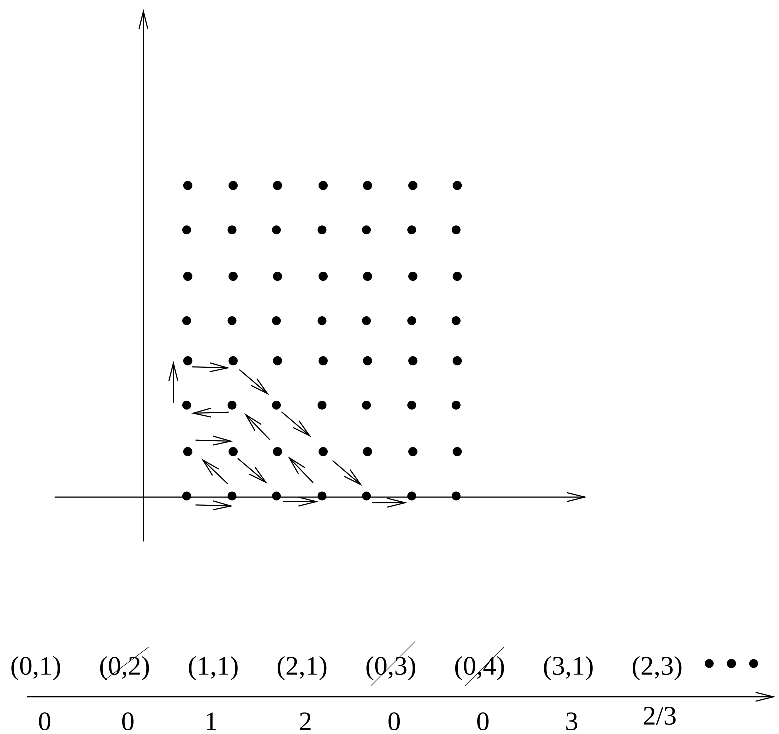


Figure 14. Counting the rational numbers

Proof. Given $\epsilon > 0$ and $y \in \mathbb{R}^{n+1}$ which is a point in the Sierpinski n-simplex, there is a mini-n-simplex with address $C = (c_1, c_2, \ldots, c_k)$ where $c_j \in \{1, 2, \ldots, n+1\}$ which is contained entirely within an ϵ neighborhood of y. The ϵ neighborhood is the set of points whose distance from y is less than ϵ. You can think of this a sort of a higher dimensional beebee (as in a beebee gun). Inside this beebee is a mini-n-simplex, and this simplex is determined by the finite sequence C.

In order for the random sequence B_A to land within this mini-sequence, we need to show that the random sequence A certainly contains an occurrence of the sequence C. *Certainty* is a strange concept in this context. It means that with probability equal to 1, there is an occurrence of the sequence C. I will not show you where the sequence C occurs, nor will I give a substantial proof. The proof will be purely existential.

When I was young, the common expression was that if you gave enough chimpanzees enough time, enough paper, and each had a typewriter, then eventually all the works of William Shakespeare would appear among the random letterings of the chimps. This was a colorful metaphor, but is relevant to the problem at hand. The complete works of Shakespeare correspond to our sequence C. We are working in more than 26-dimensions. More because we have to take into consideration punctuation marks and spaces between words. Each letter, comma, period, apostrophe, and line space corresponds to some number between 1 and 30. I do not know the length of the complete works of Shakespeare; to be safe let us estimate it as size 10^9— one billion. So in a random infinite sequence, we are trying to find a particular occurrence of a sequence of length 10^9. First we solve an easier problem.

Suppose for a moment, we are working with a random sequence taken from the set $\{1, 2, 3\}$. What is the probability that a random infinite sequence A contains the number 1? Well, the probability that first number is not 1 is 2/3. The probability that the first two numbers are not 1 is $(2/3) \times (2/3) = 4/9$. The probabilities multiply because these are independent events. The probability that among the first 15 numbers, the number 1 does not show up, then will be $(2/3)^{15} = \frac{32{,}768}{14{,}348{,}907} \approx 0.00228366$. In other words, as the sequence gets longer the probability that the number 1 *does not occur* is less than 0.3%. Therefore the probability that 1 *does occur* among the first 15 digits is more than 99.7%. Continuing along this line the probability that 1 occurs among the first m digits is $1-(\frac{2}{3})^n$. Now 2/3 is a number less than 1, and when a number less than 1 is raised to a positive integer exponent, it gets smaller. In fact, $\lim_{m\to\infty}(\frac{2}{3})^m = 0$. That is, when m gets very large the probability that 1 does not occur gets, infinitesimally small. Now we generalize this argument.

Our given sequence C — the sequence that is analogous to the collected works of Shakespeare — has length k. The numbers in the sequence are taken from among $\{1, 2, \ldots, n + 1\}$. These numbers correspond to the letters in the alphabet and punctuation marks. There are $(n+1)^k - 1$ sequences of length k that are not the sequence C. Let me illustrate this with another example. The last four digits of my social security number is some 4-digit number with digits taken from among $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$. There are $10,000 = 10^4$ such numbers. So there are $10^4 - 1$ four digit sequences that are *not* the last four digits of my social security numbers. The sequence C, albeit in principal quite long, is only one of among $(n+1)^k$ sequences. So there are $(n+1)^k - 1$ sequences of length k that *are not* C. The probability that the first k numbers of the sequence A is not C is $\frac{(n+1)^k-1}{(n+1)^k} = 1 - \frac{1}{(n+1)^k}$. The number is close to 1, but still is less than 1. Now we break the sequence A into blocks of length k. So $A = (A_1, A_2, A_3, \ldots, A_m, \ldots)$. The probability that C is not found in A_1, A_2, A_3, $\ldots$, A_m is

$$P_m = \left[1 - \frac{1}{(n+1)}\right]^m.$$

Now observe that C might have started at an arbitrary point in any one of these subsequences, and then it could overlap into one of the other sequences. So the true probability that C does not occur in the first mk numbers of A is somewhat less than the number P_m written above. On the other hand,

$$\lim_{m\to\infty} P_m = \lim_{m\to\infty} \left[1 - \frac{1}{(n+1)}\right]^m = 0.$$

So as the terms of A are enumerated, it becomes increasingly likely that C occurs. That likelihood goes to 1 as the length of the enumeration continues. This is to say, in an infinite sequence, it is certain that the given sequence C occurs.

So given an infinite number of chimpanzees with an infinite number of typewriters and infinitely many pages, it is certain that one of them will produce not only the collected works of Shakespeare, but all the computer code ever written and, heaven forbid, several copies of the current work.

In order for the sequence B to be within a beebee's width of the given point of the Seirpinkski's n-simplex, the sequence C that describes the mini-simplex that lies within the beebee's radius moust occur somewhere among the letters of A, but this happens with probability 1. It is certain that it will happen. This completes the proof. $\square$.

3.6. Fractal Dimensions. The word *fractal* was coined to indicate an object which, when its dimension is computed by traditional means, has a dimension that is not a whole number. There are many definitions of dimensions available, and on traditional objects like squares, triangles, cubes, and so forth, these definitions all agree. The definition of dimension that we use here goes as follows. A unit n-cube is defined to be the set:

$$I^n = \{(x_1, x_2, \ldots, x_n) : 0 \le x_j \le 1, \text{ for } \ j = 1, 2, \ldots, n\}.$$

So a unit 1-cube is an interval, a unit 2-cube is a square of area 1, and a unit 3-cube is a cube of volume 1. When the edge lengths are doubled the length, area, volume, or hyper-volume, increases by a factor of 2^n. That is the doubled figure can be filled with 2^n unit cubes. A 2×2 square has area 4; a $2 \times 2 \times 2$ cube has volume 8. While this idea seems tautological in the visible dimensions, getting the definition right was a major topic of research during the 20th century. We anticipate that as mathematical techniques become more sophisticated in the future, further research into the mean of dimension will be warranted.

The dimension of one of these figures, then is n. The exponent to which 2 is raised is called the *logarithm* to the base 2. If $y = 2^x$, then $x = \log_2(x) = \frac{\ln(x)}{\ln(2)}$. We use the natural logarithms (logs to the base e for computational convenience).

When the length of the edges of Sierpinski's triangle are doubled then we get exactly 3 copies of the Sierpinski triangle within it. So its dimension is $\log_2(3) \approx 1.573$. When the edges of the Sierpinski tetrahedron are doubled, there are 4 complete copies of the figure within it. Consequently, its dimension is 2. In general there are $(n+1)$-copies of the Sierpinski n-simplex within the figure when the edges are doubled. Thus the dimension of the Sierpinski n-simplex is $\log_2(n+1) = \frac{\ln(n+1)}{\ln(2)}$. These dimensions grow very slowly since they are a logarithmic function of $(n+1)$.

This completes our discussion of the chaos game, and the resulting figure which resembles the Sierpinski n-simplex.

4. Rotations

The purpose of this section is to describe rotations and projections of various figure in higher dimensional spaces. The reason that we need this discussion, is that our illustrations are in fact precisely such rotations and projections.

4.1. Rotating the Plane. The real plane $\mathbb{R}^2$ consists of ordered pairs, (x, y), of real numbers $x, y \in \mathbb{R}$. For operational reasons it is convenient to represent such an ordered pairs in columnar form. If we wish to rotate a figure in the plane through an angle θ, then we apply the matrix $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ to any point $\begin{pmatrix} x \\ y \end{pmatrix}$ in the figure. The matrix multiplication works according to the formula:

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{pmatrix}.$$

The result of rotating a triangle through such an angle is illustrated in Fig. 15.

We should make one brief comment about angle measurement. As we defined the n-simplex and projected it to the plane, we spoke of angles in terms of degrees, but henceforth we will measure angles in radians. The formula for conversion is π rad $= 180°$. So for example, an angle of $\pi/2$ is a $90°$ angle. Finally, the definition of the trigonometric functions $\cos\theta$ and $\sin\theta$ are the x and y coordinates, respectively, of the points on the unit circle that are subtended by the angle θ.

4.2. Rotating in the Cardinal Directions in Space. We can rotate a figure in 3-dimensional space about any axis whether or not that axis is a coordinate axis. Go spin a basketball. But when the axis is a coordinate axis, the rotation has a nice matrix representation. The *cardinal rotations* are defined to be the rotations that fix the lines that contain the standard coordinate directions, e_1, e_2, e_3. If the angle through which a figure is rotated is θ, then these rotations

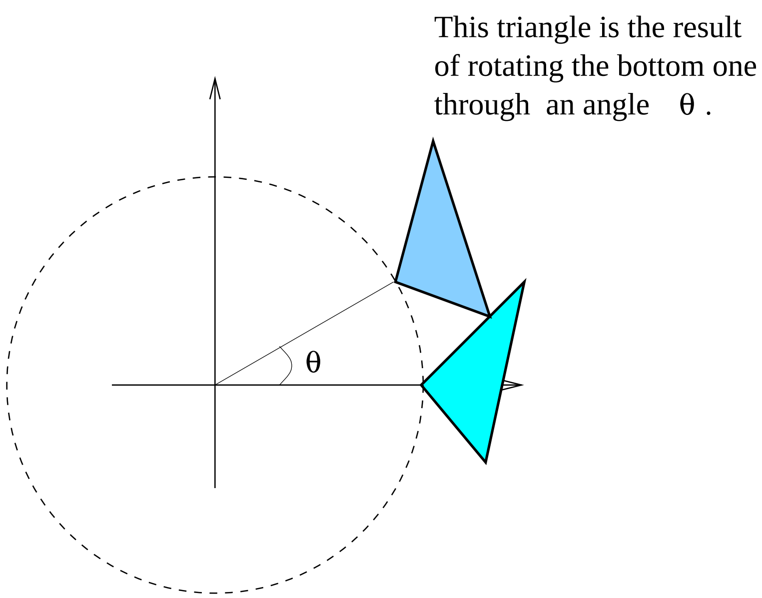


Figure 15. Rotating a triangle through an angle at the origin

are achieved via applying the matrices

$$R_3(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$R_2(\theta) = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix},$$

or

$$R_1(\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

to a point $\begin{pmatrix} x \\ y \\ z \end{pmatrix}$ in the figure. We illustrate one of these multiplications:

$$\begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \\ z \end{pmatrix}.$$

Of course, this rotation is just the same as a planar rotation, and indeed each of the rotation matrices keeps points that lie in a coordinate plane ($z = 0$, $y = 0$, or $x = 0$, respectively) within that plane.

My physicist friends always choose a coordinate system so that the axis of rotation of their basketball is the z-axis, but they are not very good basketball players. That is not to say that all physicists are bad basketball players, but those who are among my friends are not so good at basketball.

4.3. Rotations in Higher Dimensions. In general, when an $(n \times n)$-matrix is applied to a point $\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$, then the entries in a given row (say the ith row) are multiplied one-by-one to the entries x_1 through x_n, the results are added together, and the sum appears as the ith entry of the resulting vector. Formulaically, if $A = (a_{ij})_{i,j=1}^n$ then the ith entry of the resulting product, $A\vec{x}$ is $\sum_{j=1}^n a_{ij}x_j$. Or more explicitly,

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

$$= \begin{pmatrix} x_1a_{11} + x_2a_{12} + \ldots + x_na_{1n} \\ x_1a_{21} + x_2a_{22} + \ldots + x_na_{2n} \\ \vdots \\ x_1a_{n1} + x_2a_{n2} + \ldots + x_na_{nn} \end{pmatrix}.$$

In 2-dimensions, a rotation fixes a point; in 3-dimensions a rotation fixes an axis. The rotation requires a circle's worth of direction, so the size of the set that is fixed is two less than the dimension of the space in which the rotation takes place. So if we rotate a figure 4-dimensional space via a matrix like one of those above, then some plane is fixed. In general, define an (i,j) *cardinal hyper-plane*, $C_n^{i,j}$, in n-dimensional space to be the $(n-2)$-dimensional subspace which is the set

$$\{(x_1,\ldots,x_n) : x_i = x_j = 0\}.$$

In this notation C_3^{12} is the z-axis, C_3^{13} is the y-axis, and C_n^{23} is the x-axis. As in the preceding section, the notation does not become efficient until you consider higher dimensions. We will perform a rotation through an angle θ that fixes C_n^{ij} (where $1 \le i < j \le n$) in n-dimensional space via the matrix $M_{ij}(\theta)$ defined as follows:

$$\begin{array}{r} \\ i\text{th} \rightarrow \\ \\ j\text{th} \rightarrow \\ \\ \end{array} \left[\begin{array}{ccccc} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 \\ 0 & 0 & I_{j-i-1} & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 \\ 0 & \underbrace{0}_{i\text{th}} & 0 & \underbrace{0}_{j\text{th}} & I_{n-j} \end{array}\right].$$

Here the 0 entries represent blocks of various sizes each of whose entries are all 0, and the I_k represent $(k \times k)$ identity matrices; these have 1s along the diagonal and zeroes elsewhere. The identity matrices have the effect of fixing that portion of space. So this matrix fixes C_n^{ij} and rotates the plane $\{xe_i + ye_j : \ x, y \in \mathbb{R}\}$ around its origin counterclockwise through an angle θ.

The cardinal hyper-planes, C_n^{ij} in n-space are determined by pairs of whole numbers i, j with $1 \le i < j \le n$. There are $\binom{n}{2} = \frac{n(n-1)}{2}$ such pairs. As we rotate the fractal sets in the simplices, we will not have to rotate in each of these directions because an n-simplex has $(n+1)$-fold symmetry. That is an n-simplex is does not change its shape when the coordinates are permuted. In order to view the fractal sets in question from several good points of view, we will only need to rotate through the first $\lfloor \frac{n}{2} \rfloor$ cardinal directions. Here $\lfloor X \rfloor$ denotes the greatest integer less than or equal to X. So for various values:

n	1	2	3	4	5	6	7
$\lfloor \frac{n}{2} \rfloor$	0	1	1	2	2	3	3

That is, we will rotate fixing C_n^{12}, C_n^{13}, $\ldots C_n^{1k}$ where $k = \lfloor \frac{n}{2} \rfloor$. The rotations that we will depict run fully around each circle in the plane perpendicular to the cardinal hyperplane. And should we have chosen to rotate any other planes, we would see an identical figure.

Our plan, then, is to construct a fractal set via the chaos game in a higher dimensional simplex. Then we rotate the set using roughly $n/2$ different directions. Finally, we will project the rotated figure onto the plane of the paper via specific projections. So in the next subsection we discuss projections in generality.

5. Projections

Another aspect of matrices that we use is that they can be used to defined projections from one space to another. The concept of projections is familiar to a sighted person. The 3-dimensional world is projected onto the plane of the retina. So all that we with binocular vision see is two 2-dimension figures. From these, our mind creates a 3-dimensional image that is re-enforced by our sense of touch and in particular, the set of configurations that an arm with working shoulder, elbow, and wrist has.

5.1. Projections of a Cube. A cube may be projected perpendicularly so that its image is a square. This can be further projected to a line segment. Such projections are easy to achieve with matrices. For example to map the cube, to a square we can use the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

which acts on a triple (x, y, z) as follows:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

To project the cube to a line, then apply the matrix $\left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right]$ like this

$$\left[\begin{array}{ccc} 1 & 0 & 0 \end{array}\right] \left(\begin{array}{c} x \\ y \\ z \end{array}\right) = \left(\begin{array}{c} x \end{array}\right).$$

So if the cube consists of

$$\{(x, y, z) : \ 0 \leq x \leq 1, \ 0 \leq y \leq 1, \ \& \ 0 \leq z \leq 1\},$$

then the image square has the same form without a z-coordinate, and the line segment just has $x \in [0, 1]$.

The first projection is not too hard to think about. Imagine looking from far above onto a small box. The top of the box is a square, and this is all that you will see. The second projection as it applies to the box is a bit more tricky. But if all you see is a square, and you imagine that square to be a piece of paper, then look at the paper along the edge. Or alternatively, imagine the box has been crushed flat, and then look at the flat cardboard from the side. See Fig. 16

One set of figures that is illustrated seems to start from a Sierpinski triangle. As the pages flip the triangle appears more and more 3-dimensional. At a certain point it paradoxically looks homogenous, but then it continues to expand. The way this effect is achieved is by parameterizing matrices from one projection to another in a sort of unfolding. Such projections intentionally hide information. The flip book gradually reveals the information that was hidden. But I am getting ahead of myself. First, I need to show you how to project the n-simplices onto the plane in such a way that as much information as can be seen in a projection is revealed.

5.2. Projecting the n-simplex. The n-simplex, as I have constructed it, is the convex hull of the set of standard coordinate directions e_1, e_2,..., e_{n+1}. As such it lives in $(n+1)$-dimensional space. In order to see this on the page, I will have to project it to the plane. Since the n-simplex is very symmetric, I will project it to the plane so that these $(n+1)$ vertices are projected to the vertices of a regular

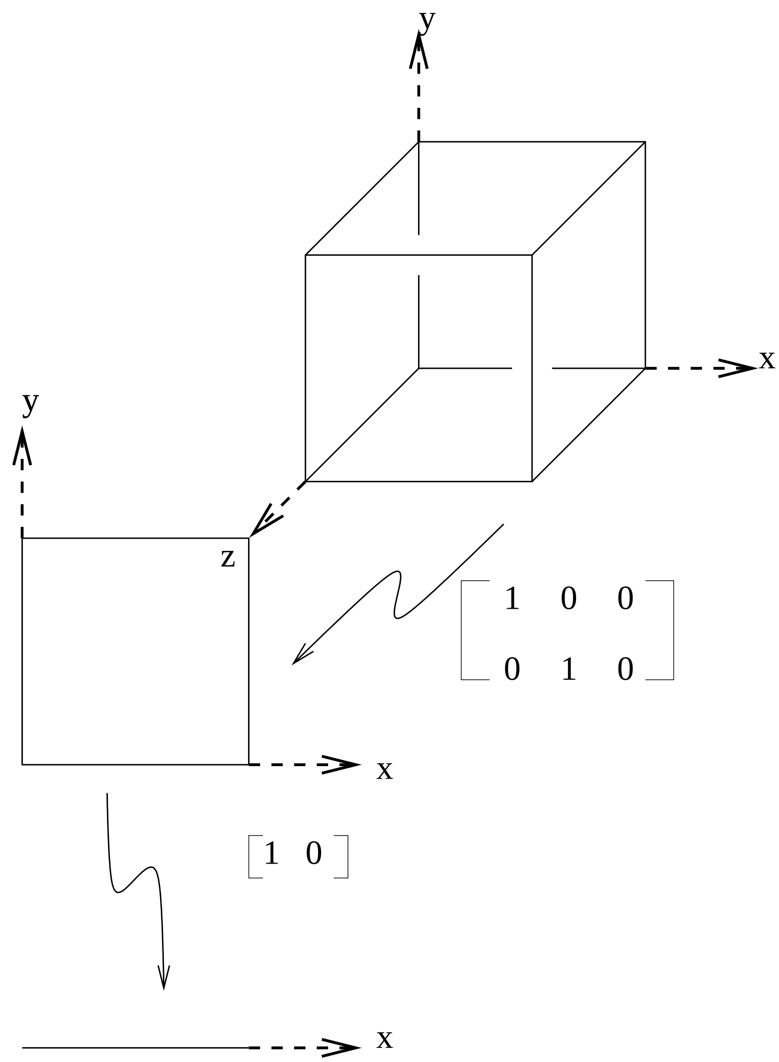


Figure 16. Projecting the cube stepwise onto a line segment

$(n+1)$ vertex polygon. The $(2 \times (n+1))$-matrix

$$\begin{bmatrix} 1 & \cos\frac{2\pi}{n+1} & \cos\frac{4\pi}{n+1} & \dots & \cos\frac{2n\pi}{n+1} \\ 0 & \sin\frac{2\pi}{n+1} & \sin\frac{4\pi}{n+1} & \dots & \sin\frac{2n\pi}{n+1} \end{bmatrix}$$

has the property that when it is applied to e_j, the resulting point is on the unit circle subtended by an angle of $2(j-1)\pi/(n+1)$. There is a shift by one index since e_1 will go to the point $(1,0)$ which is subtended by an angle 0.

There is something quite neat about matrices. When their sizes are appropriate, they may be multiplied together. So for example, a $(2 \times (n+1))$ matrix may be multiplied on the right by an $((n+1) \times (n+1))$-matrix. This product in turn may be multiplied by an $((n+1) \times 1)$-matrix (which is a point in $(n+1)$-dimensional space). For the record, when we say an $(m \times n)$-matrix we mean that the matrix has m-rows (horizontals) and n-columns (verticals). Matrices are linear functions on space, where here *linear* means that additive structure is preserved, and changes of scale are also preserved. The key for allowable multiplication is that the number of columns of the matrix on the left must match the number of rows of the matrix on the right.

Let me describe the set of transformations that I want to perform. Start with a point in $(n+1)$-dimensional space (specifically a point that is in the n-simplex and constructed via the chaos game), rotate it in one of the cardinal hyper-planes, and project the result to the plane of the picture. The point in space is given as an ordered $(n+1)$-tuple of real numbers. The rotation is achieved by an $((n+1) \times (n+1))$ matrix whose entries include the sine and cosine functions of a specific angle. The projection is achieved by a $(2 \times (n+1))$-matrix, and this is to be applied after the rotation occurs.

5.3. Turning on and off Matrices. In this portion of the text there are a couple more things that I need to describe. One is the method of gradually unfolding a singular projection as we mentioned in the last paragraph of the preceding section. The other is a method of successively rotating or unfolding the object. The key word in that sentence is *successively.*

In looking at the illustrations think of the page numbers as determining a specific time at which each event happens. In a flip book, the page number is precisely a time parameter or it is proportional to one. As I write the computer program to generate the images, I really only want one program for all of the images in each movie. So in the rotation sequence, I first rotate in the (12)-plane, then the (13)-plane, and so forth. I need to write a matrix that will turn on a given rotation on a specific interval of time.

A *characteristic function* is a function defined on any set to take the value 1 at any point in the set and to take the value 0 outside of the set. The characteristic function on a closed interval of numbers $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ is the function defined on the line by the formula

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

The characteristic function $\chi_{[a,b)}$ is defined in a manner analogous to the function above except at b it takes the value 0. The parenthesis ")" indicates that the interval in question does not include the right end-point; a left parenthesis "(" would indicate that the left end-point is deleted.

Look at yourself. Your characteristic function defines you in your space. It is exactly 1 on that part of the world that is you and it is 0 everywhere else. John Lennon's (and Paul MacCartney's?) characteristic function is a bit different than yours: "I am he as you are he as you are me and we are all together," but it is the same as the Walrus's.

When a matrix is multiplied by a number, all the entries in the matrix are multiplied by that number. For example, if I want a given matrix to act on space in a given interval of time, but not act at other values of time, I can multiply the matrix by the characteristic function of that interval. Suppose for example that you have two matrices A and B and each depends on a parameter t whose value ranges from 0 to 1. You want A to act when $0 \leq t < 1/2$ and you want B to act when $1/2 \leq t \leq 1$. Then form the matrix

$$\chi_{[0,1/2)}(t)A + \chi_{[1/2,1]}(t)B$$

where the corresponding entries of the matrices are added entry-by-entry.

There is another trick that is needed. Each of the rotation matrices $M_{ij}(\theta)$ depends on the parameter θ which ranges between 0 and

2π. I want to apply M_{12}, then M_{13} and so forth, but I want these to happen successively. So I need to change the parameterizations so that in one interval the first matrix acts with the parameter running around a full cycle, and then the next matrix acts with the parameter running around a full cycle. The trick is to alter the speed of parameterizations. For example, if you have 6-matrices, $A_1(t), \ldots, A_6(t)$ each depends on the parameter t and you want them to act successively over the interval $t \in [0, 1]$, then form the matrix sum

$$\chi_{[0,\frac{1}{6})} A_1(6t) + \chi_{[\frac{1}{6},\frac{2}{6})} A_2(6t-1) + \chi_{[\frac{2}{6},\frac{3}{6})} A_3(6t-2)$$

$$+\chi_{[\frac{3}{6},\frac{4}{6})} A_4(6t-3) + \chi_{[\frac{4}{6},\frac{5}{6})} A_5(6t-4) + \chi_{[\frac{5}{6},1]} A_6(6t-5).$$

Notice that the formula is easier to understand when the fractions are not reduced. Also, observe that when $t = j/6$, the function $6t - j = 6(j/6) - j = 0$ while at $(j+1)/6$, $6t - j = 6((j+1)/6) - j = 1$ and each value from 0 to 1 is achieved. Finally, you can adjust for $\theta \in [0, 2\pi]$ by multiplying all of the arguments $6t - j$ by 2π.

Let me give an example. In 5-space I want to rotate the 4-simplex fixing the 3-dimensional space, C_5^{12}, then I want to rotate fixing the 3-space C_5^{13}. I form the sum

$$\chi_{[0,\frac{1}{2})} M_{12}(4\pi t) + \chi_{[\frac{1}{2},1]} M_{13}(2\pi(2t-1)).$$

Let me detail how this formula works. When $t \in [0, 1/2)$ the first characteristic function returns a value of 1 while the second returns a value of 0. So during this period of time, only the matrix $M_{12}(\theta)$ is acting. Let $\theta = 4\pi t$. Then as t ranges from 0 to 1/2, the angle θ ranges from 0 to 2π. So the full rotation M_{12} is implemented in this first half-interval of time. When $t \in [1/2, 1]$, the second characteristic function returns 1 while the first returns 0. So the second matrix $M_{13}(\theta)$ is applied. Letting $\theta = 2\pi(2t - 1)$, you see at $t = 1/2$, the angle $\theta = 0$, and at $t = 1$, the angle $\theta = 2\pi$. The only peculiarity is the interface at $t = 1/2$; literally, the first rotation never completes. But practically, that won't really matter because the program takes a sequence of values rather than a continuum, and I can get the sequence as close to 1/2 from the left-hand-side as I need to simulate motion.

5.4. Sequentially Unfolding. Just as the cube could be projected to a line segment in the crushed box metaphor, the n-simplex can be projected to a line segment or a triangle. The projection to a segment may not be so interesting, so I start by projecting to a triangle. Then I parameterize the projection so that it moves from the projection onto a triangle to a projection onto a square, then to a pentagon and so forth. The situation is schematized in Fig. 17. All but two vertices of the n-simplex are projected to the same point of a triangle. During the first interval of time the vertices move to the vertices of a square. Three vertices of the simplex are mapped to three corners of the square and the rest get mapped to the fourth corner. During the next period of time, the vertices move from those of a square to a pentagon with all but four vertices mapping to a single vertex of the pentagon. The process can continue until the entire n-simplex is unfolded onto a regular $(n+1)$ vertex polygon.

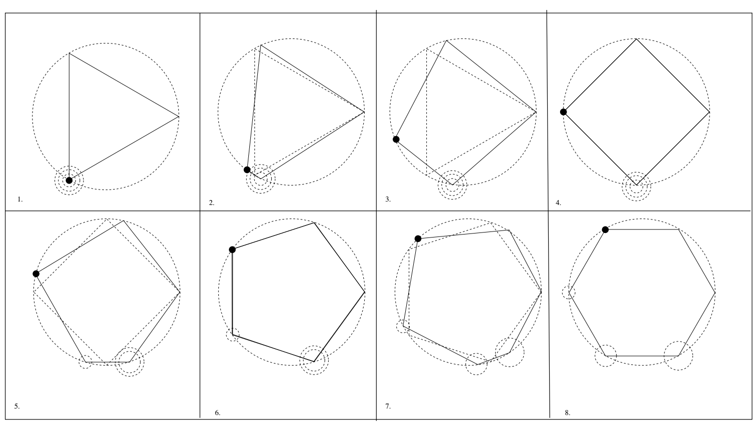


Figure 17. Schematically unfolding the projection of an n-simplex

I will describe this process in mathematical detail. Consider the $(2 \times (n+1))$ matrix:

$$A_3 = \left[\begin{array}{ccc} 1 & \cos\frac{2\pi}{3} & \underbrace{\begin{array}{cccc} \cos\frac{4\pi}{3} & \cos\frac{4\pi}{3} & \dots & \cos\frac{4\pi}{3} \\ \sin\frac{4\pi}{3} & \sin\frac{4\pi}{3} & \dots & \sin\frac{4\pi}{3} \end{array}}_{(n-1) \text{ columns}} \\ 0 & \sin\frac{2\pi}{3} & \end{array} \right].$$

This matrix, A_3 maps e_1 onto $(1,0)$, and e_2 goes to a point offset by an angle of $2\pi/3 = 120°$. The remaining vectors $e_3, \ldots, e_{n+1}$ all get mapped to the point on the unit circle at an angle of $4\pi/3 = 240°$.

Now for any two points (a,b) and (c,d) in the plane the line segment that joins them is given by the formula $(1-t)(a,b)+(t)(c,d) = ((1-t)a+(t)c,(1-t)b+(t)d)$. In other words, when $t=0$ the formula gives (a,b), and when $t=1$, the formula gives (c,d). For other values of $t \in [0,1]$, the formula gives a point on the line joining these two points. This line segment is precisely the convex hull of these two points. The points in the plane that I am interested in are the columns of the matrix A_3 and the columns in the matrix A_4 where

$$A_4 = \left[\begin{array}{ccc} 1 & \cos\frac{\pi}{2} & \cos\pi \\ 0 & \sin\frac{\pi}{2} & \sin\pi \end{array} \underbrace{\begin{array}{cccc} \cos\frac{3\pi}{2} & \cos\frac{3\pi}{2} & \ldots & \cos\frac{3\pi}{2} \\ \sin\frac{3\pi}{2} & \sin\frac{3\pi}{2} & \ldots & \sin\frac{3\pi}{2} \end{array}}_{(n-2)\text{ columns}}\right].$$

So I construct the parameterized matrix

$$A_{3\to4}(t) = \left[\begin{array}{ccc} 1 & (1-t)\cos\frac{2\pi}{3}+t\cos\frac{\pi}{2} & (1-t)\cos\frac{4\pi}{3}+t\cos\pi \\ 0 & (1-t)\sin\frac{4\pi}{3}+t\sin\frac{\pi}{2} & (1-t)\sin\frac{4\pi}{3}+t\sin\pi \end{array}\right.$$

$$\left.\begin{array}{ccc} (1-t)\cos\frac{4\pi}{3}+t\cos\frac{3\pi}{2} & \ldots & (1-t)\cos\frac{4\pi}{3}+t\cos\frac{3\pi}{2} \\ (1-t)\sin\frac{4\pi}{3}+t\sin\frac{3\pi}{2} & \ldots & (1-t)\sin\frac{4\pi}{3}+t\sin\frac{3\pi}{2} \end{array}\right].$$

The matrix $A_{3\to4}(t)$ fixes the image of e_1 to be $(1,0)$; it moves the image of e_2 from a point at an angle of $120°$ to a point at an angle of $90°$. The vector e_3 moves from $240°$ to a point at $180°$, and the remaining basis vectors move to the point three fourths of the way around the circle. At $t=1$, you get

$$A_{3\to4}(1) = \left[\begin{array}{ccccccc} 1 & 0 & -1 & 0 & 0 & \ldots & 0 \\ 0 & 1 & 0 & -1 & -1 & \ldots & -1 \end{array}\right].$$

Next, the image of an n-simplex opens from a square to a pentagon in during the interval of time $t \in [1,2]$. But before it does, I remind you that the motions can be controlled by means of characteristic functions. That is I will turn on the matrix $A_{3\to4}(t)$ for $0 \le t < 1$, and then turn on $A_{4\to5}(t)$ for $1 \le t \le 2$.

As you can tell from the expression for $A_{3\to4}(t)$, the matrix is getting too crowded to fit on one typeset line. There is a standard notation that we will adopt that will allow me to write this matrix more succinctly. First, suppose ζ_{n+1} is the point on the unit circle subtended by the angle $\frac{2\pi}{n+1} = 360°/(n+1)$. For example, $\zeta_5 = (\cos\frac{2\pi}{5}, \sin\frac{2\pi}{5})$. We think of this as a column vector, but write it as a row vector. Since each of the points on the $(n+1)$-gon is subtended by an angle that is a multiple of ζ_{n+1}, we write the point subtend by an angle of $j\left(\frac{2\pi}{n+1}\right)$ as ζ_{n+1}^j for $j = 1, 2, \ldots, n+1$. The notation makes good sense when you think of a point in the plane as a complex number. The points on the unit circle in the complex plane are those points whose distance from the origin is 1. Multiplication in the complex plane by such a point is just a rotation through the corresponding angle. So the addition of angles can be written in multiplicative notation via the use of the complex numbers.

In this new notation the unfolding $A_{3\to4}(t)$ can be written as

$$A_{3\to4}(t) = \big((1-t)\zeta_3^0 + t\zeta_4^0, (1-t)\zeta_3^1 + t\zeta_4^1,$$

$$(1-t)\zeta_3^2 + t\zeta_4^2, (1-t)\zeta_3^2 + t\zeta_4^3, \ldots, (1-t)\zeta_3^2 + t\zeta_4^3\big).$$

And even though the formula does not fit on one line (well the first coordinate is still always $(1,0)$), it has the advantage that it will be easy to generalize to unfolding a polygon with n-vertices to one with $(n+1)$-vertices.

So I will unfold an n-simplex projected onto a triangle to one projected onto a square during the first interval of time, that is when $t \in [0,1)$. Then during the time interval $t \in [1,2)$, I will unfold the n-simplex projected onto a square to be projected onto a pentagon. During the interval $t \in [2,3)$, it will unfold from pentagon to hexagon, and so forth. I can achieve this sequence of unfoldings as the sum,

$$\chi_{[0,1)}A_{3\to4} + \chi_{[1,2)}A_{4\to5} + \ldots$$

$$+\chi_{[u-3,u-2)}A_{u\to(u+1)} + \ldots + \chi_{[n-3,n-2)}A_{(n-1)\to n}.$$

In order to derive the general formula for $A_{u\to(u+1)}$, examine the case $A_{4\to 5}$. We have,

$$\begin{aligned} A_{4\to 5}(t) \quad = \quad & \big((2-t)\zeta_4^0 + (1-t)\zeta_5^0, \\ & (2-t)\zeta_4^1 + (1-t)\zeta_5^1, \\ & (2-t)\zeta_4^2 + (1-t)\zeta_5^2, \\ & (2-t)\zeta_4^3 + (1-t)\zeta_5^3, \\ & (2-t)\zeta_4^3 + (1-t)\zeta_5^4, \\ & (2-t)\zeta_4^3 + (1-t)\zeta_5^4, \ldots, \\ & (2-t)\zeta_4^3 + (1-t)\zeta_5^4\big) \end{aligned}$$

where the number of terms of the form $(2-t)\zeta_4^3 + (1-t)\zeta_5^4$ is $n-3$ since we are unfolding the image of the n-simplex, this involves $n+1$ separate coordinates, and the first 4 are of a different form. In general,

$$\begin{aligned} A_{k\to(k+1)}(t) \quad = \quad & \big(((k+1)-t)\zeta_k^0 + (k-t)\zeta_{k+1}^0, \\ & ((k+1)-t)\zeta_k^1 + (k-t)\zeta_{k+1}^1, \ldots, \\ & ((k+1)-t)\zeta_k^{(k-1)} + (k-t)\zeta_{k+1}^{(k-1)}, \\ & ((k+1)-t)\zeta_k^{(k-1)} + (k-t)\zeta_{k+1}^{k}, \\ & ((k+1)-t)\zeta_k^{(k-1)} + (k-t)\zeta_{k+1}^{k}, \ldots, \\ & ((k+1)-t)\zeta_k^{(k-1)} + (k-t)\zeta_{k+1}^{k}\Big). \end{aligned}$$

Finally, work through each of the matrices $A_{k\to(k+1)}$ successively by multiplying by characteristic functions. For $t \in [0, n-2]$, let

$$M(t) = \sum_{j=1}^{n-3} \chi_{[j-1,j)} A_{(j-1)\to j}(t) + \chi_{[n-3,n-2]} A_{(n-3)\to(n-2)}(t).$$

6. Multinomial Coefficients

There is a lovely correspondence between the Sierpinski triangle and the points of Pascal's triangle when the latter are colored black or white depending upon their *parity*: oddness or even-ness. Pascal's triangle is the triangle that is formed from the binomial coefficients $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. These numbers are either odd or even. There is a way of arranging the numbers in Pascal's triangle, so that when odd numbers are colored black, and even numbers are colored white, the resulting figure looks like Sierpinski's triangle. I will explain this more thoroughly below.

There are also things called multinomial coefficients that can easily be arranged in the shape of an n-simplex. When these are colored black and white, then it simulates Sierpinski's n-simplex. This fact is probably obvious to the experts, but I only have glimpses of it having been explicitly observed. My proof will involve a nice combination of elementary results about divisibility, and some geometric tricks.

The current section may be the mathematically most technical, but it is also the end of the chapter. It represents the part of the discussion that is new, but the discussion here is not necessary for the description of the figures; I finished that in the section above. It does, however, follow my discussion of beauty and romanticism. An observation about Pascal's triangle that is well known to many, can be easily generalized to higher dimensional analogues. The beauty that is apparent to many is transcendental and extends to realms in which it cannot be experienced directly.

6.1. Binomial Coefficients. I expect that you have seen the binomial theorem:

$$(x+y)^r = \sum_k^r \binom{r}{k} x^k y^{r-k}.$$

The binomial coefficients $\binom{r}{k} = \frac{r!}{k!(r-k)!}$ count, the number of ways that a k element subset can be chosen from an r element set. By definition $0! = 1$, and $r! = r(r-1)!$ The binomial coefficients can be arranged in the shape of a triangle, Pascal's triangle. For my purposes, the arrangement will be the most convenient if the coefficient $\binom{r}{k}$ is located at the point $(k, r-k)$ in the plane. Then the miracle of Pascal's triangle is that the item at position $(k, r+1-k)$ can be obtained as the sum of the elements at position $(k, r-k)$ and $(k-1, r-k-1)$. That means that the arrangement is made in such a way that in order to compute the value at a particular location, then you look to the left and down and add these two values. The array below illustrates the process. Along the horizontal axis the values

$1 = \binom{r}{r}$ appear, and along the vertical the values $1 = \binom{r}{0}$ appear. In the location, $(3,5)$ the value $\binom{8}{3} = 56$ appears. Remember that the bottom row and the left column are counted as number 0; they correspond to the coordinate axis. I used vertical dots to indicate that the pattern continues in a vertical fashion. It is somewhat easier to add the entries to the left and below the given entry in this way.

$$
\begin{array}{ccccccccc}
1 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 8 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 7 & 28 & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 6 & 21 & 56 & \vdots & \vdots & \vdots & \vdots & \vdots \\
1 & 5 & 15 & 35 & 70 & \vdots & \vdots & \vdots & \vdots \\
1 & 4 & 10 & 20 & 35 & 56 & \vdots & \vdots & \vdots \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & \vdots & \vdots \\
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \vdots \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1
\end{array}
$$

The ease in which addition can occur is not the reason that I arranged the elements in Pascal's triangle this way. The real reason is that I want to show that when the elements in Pascal's triangle are reduced to be odd (black) or even (white), then the resulting figure (for the infinite Pascal's triangle) resembles Sierpinski's triangle. There will be some more work to do to create the resemblance, and I will do the work for the multinomial coefficients all at once. The work that is involved requires two steps. First, I have to rescale the infinite triangle so that it fits into the triangle $\{(x,y) : x, y \geq 0, \ \& \ y \leq 1-x\}$. Then I have to map that triangle upward to the equilateral triangle in space which is the convex hull of the three coordinate vectors $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$. Everything that is said for triangles in this paragraph will apply equally to n-simplices in the general case.

6.2. Multinomial Coefficients. A *monomial* in n-variables, x_1, x_2, …, x_n is an expression of the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ where the exponents $k_1, \ldots, k_n \in \{0, 1, 2, \ldots\}$. The *degree*, r, of the monomial is the sum of the exponents $r = k_1+k_2+\cdots+k_n$. Consider the coefficients of the various monomials in the expansion of $(x_1+x_2+\cdots+x_n)^r$. These are called the *multinomial coefficients.* The binomial coefficients are a special case of these ($n = 2$), and the multinomial coefficients are given by the formula

$$\binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \frac{(k_1 + k_2 + \cdots + k_n)!}{k_1!k_2! \cdots k_n!}.$$

In this notation, the binomial coefficient is given as

$$\binom{r}{k} = \binom{r}{k, r-k}.$$

A *recursion relation* among a sequence of variables is a relationship in which the values of the current element of the sequence are determined by the previous values of the sequence. *Pascal's recursion* is given by the formula:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

This relationship among the binomial coefficients is how most of us compute the small values; it is how the triangle above is constructed: The value at position $(k, r-k)$ is determined as the sum of the values of the coefficient below (at $(k, r-k-1)$) and to the left (at $(k-1, r-k)$). Observe, $r-1 = k+(r-k-1) = (k-1)+(r-k)$, so that the values at these coordinates correspond to the binomial coefficients in the recursion relation.

There is a recursion relation that is available for the multinomial coefficients. In its arithmetic form it is given as

$$\binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \sum_{i=1}^{n} \binom{k_1 + k_2 + \cdots + (k_i - 1) + \cdots + k_n}{k_1, k_2, \ldots, (k_i - 1), \ldots, k_n}.$$

The proof of this recursion is a very clever application of the technique of adding fractions:

$$\begin{aligned}
&\sum_{i=1}^{n} \frac{(k_1 + k_2 + \cdots + (k_i - 1) + \cdots + k_n)!}{k_1!k_2!\ldots(k_i-1)!\ldots k_n!} \\
&= \frac{(k_1 + k_2 + \cdots + k_i + \cdots + k_n - 1)!}{(k_1-1)!(k_2-1)!\ldots(k_i-1)!\ldots(k_n-1)!} \cdot \\
&\quad \cdot \sum_{i=1}^{n} \frac{1}{k_1 k_2 \cdots k_{i-1} k_{i+1} \cdots k_n} \\
&= \frac{(k_1 + k_2 + \cdots + k_i + \cdots + k_n - 1)!}{(k_1-1)!(k_2-1)!\ldots(k_i-1)!\ldots(k_n-1)!} \cdot \\
&\quad \cdot \sum_{i=1}^{n} \frac{k_i}{k_1 k_2 \cdots k_n} \\
&= \frac{(k_1 + k_2 + \cdots + k_i + \cdots + k_n)!}{k_1!k_2!\ldots k_i!\ldots k_n!}
\end{aligned}$$

In n-dimensional space $\mathbb{R}^n$ imagine the lattice or scaffold that consists of those points whose coordinates are non-negative integers. In the plane it is a waffle pattern, in 3-space it is a cubical complex that emanates in the non-negative region of space. In general, it is the region of non-negative space in which the coordinates are all natural numbers.

I introduce a notation, that helps us keep track of the multinomial coefficients. Let

$$[k_1, k_2, \ldots, k_n] = \binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n}.$$

Now label the point $(k_1, k_2, \ldots, k_n)$ with the multinomial coefficient $[k_1, k_2, \ldots, k_n]$. The meaning of the recursion relation above is that in order to compute the value of the coefficient at $(k_1, k_2, \ldots, k_n)$, then you can look at the adjacent points a unit distance behind the given point (behind in each and every direction) and add these values together. To initialize the recursion, label the points on the axes $(k_1, 0, 0, \ldots, 0)$, $(0, k_2, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, k_n)$ with 1s. And label the points in the planes at which one coordinate (the ith coordinate) is 0 with the multinomial coefficients $[k_1, k_2, \ldots, k_{i-1}, k_{i+1}, \ldots, k_{n-1}]$.

The multinomial coefficients are symmetric. That is they have the same values no matter how the k_is are permuted. This symmetry will be used in the Theorem below to simplify the proof.

The labeling process and its recursion is quite remarkable in my opinion. Even better, the recursion relation will help me prove the principal result of this section. Namely, when a color of black is assigned to the multinomial coefficients that are odd, and a color of white is assigned to those that are even, and the multinomial coefficients are rescaled to the n-simplex, then the resulting figure resembles a Sierpinski n-simplex.

In order to create a *Pascal's n-simplex*, I truncate the non-negative lattice along the region at which the sum of the k_js is constant. The region bounded by the hyper-planes $k_i = 0$ for $i = 1, 2, \ldots, n$ and $\sum_{i=1}^{n} k_i = r$ forms a right rectangular n-simplex. This is not the simplex that we have been identifying with the standard n-simplex, but we will be able to map this simplex orthographically up to the standard one after it is rescaled. In the description of the tetrahedron, I discussed the northwestern tetrahedron in the room in which I was sitting and that I asked you to imagine. This tetrahedron of volume $1/6$ is bounded by the coordinate planes and the plane $x+y+z = 1$. If I move this diagonal plane to the plane $x+y+z = r$, where $r > 1$, then a piece of Pascal's 3-simplex fits within this tetrahedral region. The trinomial coefficients in this region are those in the expansion of $(x_1 + y_1 + z_1)^s$ where $s \leq r$. I am getting a little bit ahead of myself though. First, I want to examine the parity of the multinomial coefficients.

Theorem 6.1. *Suppose that $k_1, k_2, \ldots, k_n < 2^j$. Then*

(1) for any $i = 1, 2, \ldots, n$, 2 divides the difference

$$[k_1, \ldots, k_{i-1}, k_i + 2^j, k_{i+1}, \ldots, k_n] - [k_1, \ldots, k_i, \ldots, k_n].$$

(2) Moreover for any $\ell \geq 2$, the multinomial coefficient

$$[k_1 + 2^j, k_2 + 2^j, \ldots, k_\ell + 2^j, k_{\ell+1}, \ldots, k_n]$$

is an even number.

Proof. Because of the symmetry in the multinomial coefficients, it suffices to prove statement (1) in the case that $i = 1$. In fact, this case is no more simple than the general case. More importantly, statement (2) implies that whenever two or more of the coordinates have a power of two added, then the resulting multinomial coefficient is an even number. That is, it does not matter which of the ℓ places the powers of two appear, they may be consecutive or not. What matters is that two of the coordinates exceed the others by a power of 2 and that this power of 2 is larger than any of the k_i. I will discuss the geometric significance subsequently.

The proof is by induction on n. First establish the result for $n = 1$ which is the most trivial case. Then examine the case when $n = 2$. This case is well known since it is the case of Pascal's triangle. But the proof in this case will indicate the general case.

When $m = 1$, the multinomial coefficient is not multi; It is singular: $[k_1] = \frac{k_1!}{k_1!} = 1$. Thus statement (2) does not apply. On the other hand, every value is 1, and therefore is odd. The difference of any two values then is 0 which is an even number.

In case $n = 2$, begin by examining the value $[1, 1]$. This case, $[1, 1] = [0 + 2^0, 0 + 2^0]$ is a baby case for property (2). That is, $j = 0$ is the first value for which we can study property (2). When $j = 0$, the only number that is less than $2^0 = 1$ is 0. Now by the Pascal recursion $[1, 1] = [1, 0] + [0, 1]$. Furthermore, $[1, 0] = 1$ and $[0, 1] = 1$ from the case $n = 1$ because both $[k_1, 0] = 1$ and $[0, k_2] = 1$ as long as k_1 and k_2 are non-negative. So we have that $[1, 1]$ is even.

The preceding paragraph is a little long winded, but it is meant to indicate how we can obtain a given value from (A) the values with smaller numbers, and (B) the values along the borders of the regions in the Pascal triangle. In the multinomial case, we will need all of the the information that is "below" the given coefficient.

Suppose that the exponent $j = 1$. Then we are interested in the parity (oddness or evenness) of the differences between the binomial coefficients $[2, 0]-[0, 0]$, $[2, 1]-[0, 1]$, $[3, 0]-[1, 0]$, and $[3, 1]-[1, 1]$. We get $[2, 0]-[0, 0] = 0$ since these are the 1nomial coefficients. Similarly, $[3, 0]-[1, 0] = 0$. To compute the difference $[2, 1]-[0, 1]$, first compute, using the Pascal recursion, $[2, 1] = [1, 1] + [2, 0]$. Then $[2, 1] - [0, 1]$ is even because $[2, 0] = 1$ and $[1, 1]$ is even. The recursion $[3, 1] = [2, 1] + [3, 0]$ and the previous computations allow us to conclude that

$[3,1]$ is even: *vis a vis* $[2,1]$ is odd and $[3,0]$ is odd. Since $[3,1]$ and $[1,1]$ are even, then we conclude that the difference is even.

Let me recapitulate. Any given value can be computed by Pascal's triangle and the values for which are below the given location. Specifically, suppose that we have shown that for all k_1 and k_2 such that $k_1 + k_2 < r$, that $[k_1 + 2^j, k_2] - [k_1, k_2]$ and $[k_1, k_2 + 2^j] - [k_1, k_2]$ are even. Suppose further that $[k_1 + 2^j, k_2 + 2^j]$ is even, and that $k_1, k_2 < 2^j$. Use the inductive hypothesis and Pascal's recursion,

$$[(k_1+1)+2^j, k_2] - [k_1, k_2] = [k_1+2^j, k_2] + [k_1+1+2^j, k_2-1] - [k_1, k_2].$$

But the first two terms on the right have the same parity as

$$[k_1, k_2] + [k_1 + 1, k_2 - 1] = [k_1, k_2].$$

Thus the difference on the left is even. This trick works provided that neither $k_1 = 0$, nor $k_2 = 0$. But these cases follow from the 1-nomial case. Now we examine the "diagonal" situation.

$$[(k_1 + 1) + 2^j, k_2 + 2^j] = [k_1 + 2^j, k_2 + 2^j] + [k_1 + 1 + 2^j, k_2 - 1 + 2^j].$$

By the inductive hypothesis, the two terms on the right are both even. Thus the term on the left is even. Thus the result of the theorem hold when $r = 2$.

We assume the result of the theorem in the case $n - 1$ to show that this case implies the case n.

We begin by examining the parity of $[(k_1 + 1) + 2^j, k_2, \ldots, k_n]$ when $1 + \sum_{i=1}^n k_i < 2^j$. By the recursion relation above, we have

$$[k_1 + 1 + 2^j, k_2, \ldots, k_n] = [k_1 + 2^j, k_2, \ldots, k_n] + \sum_{i=2}^{n} [(k_1 + 1) + 2^j, k_2, \ldots, k_i - 1, \ldots k_n].$$

The $i = 1$ term of the sum has a different flavor than the other terms. By induction on $\sum_{i=1}^n k_i = r$ each term on the right has the same parity as the term with the 2^j missing. Thus,

$$[k_1 + 2^j, k_2, \ldots, k_n] + \sum_{i=2}^{n} [(k_1 + 1) + 2^j, k_2, \ldots, k_i - 1, \ldots k_n] \equiv_2 [k_1, k_2, \ldots, k_n]$$

$$+\sum_{i=2}^{n}[(k_1+1),k_2,\ldots,k_i-1,\ldots k_n]$$

$$=[(k_1+1),k_2,\ldots,k_n].$$

Here we are using the somewhat common notation: $\equiv_2$ means that two numbers have the same parity or equivalently that their difference is an even number. Notice that it does not matter where we add the 1 nor where we add the power of 2. The inductive hypothesis on n is used in case any one of the k_js is 1. In that case the terms on the right hand side, are of the form of multinomial coefficients with $n-1$ terms.

Next we examine statement (2). The quantity $[(k_1+2^j,k_2+2^j,\ldots,k_\ell+2^j,\ldots,k_n]$ can be written in terms of the recursion relation. If any $k_i=1$, then by induction on n, the terms on the right of the recursion relation that involve 0 or $0+2^j$ are even. The remaining terms are also even by induction on $\sum_{i=1}^{n}k_i=r$. If none of the $k_i=1$, then the induction only depends on this last sum. This completes the proof. □

6.3. Rescaling the Pascal n-simplex. I want to fit the multinomial coefficients into the shape of the standard n-simplex. There are two steps; this section defines the first step.

Before I start, though I need a name for the set

$$\{(x_1,x_2,\ldots,x_n):\sum_{i=1}^{n}x_i\le 1,\quad \&\quad x_i\ge 0 \text{ for } i=1,2\ldots,n\}.$$

We describe this set as the right isosceles n-simplex of volume $1/n!$. It is right because the coordinate facets meet each other at right angles. It is isosceles because the $(n-1)$-simplicial facet have edges that are of equal length. The volume calculation follows a similar inductive argument as in Section2.4. This set needs a name so, for no particular reason (other than it is a family name) let us call it *Eleanor.*

The parity property of Theorem 6.1 allows us to color the lattice points in the non-negative region of n-space to be either black (when the multinomial coefficient is odd) and white (when the multinomial coefficient is even).

First we consider the multinomial coefficients for which $k_1 + k_2 + \ldots + k_n \leq 1$. Naturally, either all $k_i = 0$, or all but one of the k_is is 0 and the remaining $k_i = 1$. Thus we can color the origin in n-space, and the unit coordinate points e_i black.

Now I consider the multinomial coefficients when $\sum_{i=1}^{n} k_i \leq 3$. These are arranged on the non-negative lattice within the region $\sum_{i=1}^{n} x_i \leq 3$. We divide each of the coordinates by 3, and refit the black and white picture into the region for which $\sum_{i=1}^{n} x_i \leq 1$. The result of the parity theorem is that those points inside the right isosceles n-simplex Eleanor that were originally colored black (the origin and the coordinate points e_i) are still colored black.

Next I consider the multinomial coefficients for which $\sum_{i=1}^{n} k_i \leq 7$. Color them black or white depending on their parity. Then rescale by dividing each position by 7. In this way, once again the figure fits into a region in which all the coordinates add to be less than 1. The Parity Theorem 6.1 again ensures that those points in Eleanor after the rescale that were black remain black.

The process continues. Having rescaled the black and white colored region, that corresponds to the parity of the multinomial coefficients $\sum_{i=1}^{n} k_i \leq 2^j - 1$ by a factor of $1/(2^j - 1)$, we obtain a figure in the region of n-space for which $\sum_{i=1}^{n} x_i \leq 1$. This figure has the property that its points are either colored white or black. We only consider the black points. Then we shrink the black and white multinomial coefficients for which $\sum_{i=1}^{n} k_i \leq 2^{j+1} - 1$ by a factor of $1/(2^{j+1} - 1)$. The points inside Eleanor at the previous stage that were black remain black, but more black points are added to the inside of Eleanor.

Finally, consider this a result of an infinite process. So many of the points in the Eleanor region of space will be colored black or white that the Eleanor region will appear to be the projection of the Sierpinski n-simplex under the projection from $(n+1)$-space to n-space in which the last coordinate is mapped to 0. (Write down a matrix for this projection). Let us reuse the name *Eleanor* to mean those points in the right isosceles n-simplex

$$\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i \leq 1, \quad \& \quad x_i \geq 0 \text{ for } i = 1, 2 \ldots, n\}$$

that are colored black or white via the limit of the sequence of the contractions of the black and white colored Pascal's n-simplex.

There is another important property of this figure that we need to discuss. At the very first stage only the $(n+1)$ points $(0, 0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $(0, 1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 0, 1)$ are colored black. But the Parity Theorem 6.1, means that this initial pattern of black is duplicated in each of the coordinate directions. We will discuss this aspect in a little more detail in the next subsection.

6.4. Pitching the Tent. We can map the right isosceles n-simplex

$$\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i \leq 1, \quad \& \quad x_i \geq 0 \text{ for } i = 1, 2 \ldots, n\}$$

that contains the Eleanor figure to the standard n-simplex, that is to the subset of $(n+1)$-space which is the convex hull of the set of unit coordinate vectors $B_n = \{e_1, e_2, \ldots, e_{n+1}\}$. The function

$$(x_1, x_2, \ldots, x_n) \mapsto (x_1, x_2, \ldots, x_n, 1 - \sum_{i=1}^{n} x_i)$$

does the trick. By definition, all of the coordinates on the right add up to 1; we call this trick *pitching a tent* and the function itself is called *the tent map.* In the 2-dimensional case, the isosceles right triangle in the plane is stretched onto the equilateral triangle in space. Under this transformation, the right isosceles n-simplex that has walls intersecting at right angles is mapped to the standard equilateral n-simplex in $(n+1)$-space. Furthermore, the origin gets mapped to the point $e_{n+1} = (0, 0, \ldots, 1)$. The e_js for $j \leq n$ get mapped to e_js for which the last coordinate is 0.

We examine the image of the black points under the tent map. As in the previous paragraph, the origin in n-space gets mapped to the point e_{n+1} while the coordinate vectors in n-space get mapped to their counter-parts in $(n+1)$-space. These $(n+1)$ black points in Pascal's n-simplex form the basis by which we build the entire configuration.

The construction of the Eleanor simplex in n-space required us to first plot the $(n+1)$ points $(0, 0, \ldots, 0)$, $(1, 0, \ldots, 0)$, $\ldots$, $(0, 0, \ldots, 1)$ (that is color them black), and subsequently shrink them and copy the shrunken figure in each of the coordinate directions. This process goes on infinitely and resembles the process for constructing Sierpinski's triangle. Under the tent map, the black regions of the Eleanor figure get mapped to the Sierpinski triangle. That is the construction of the binary Pascal's n-simplex (Eleanor) consists of starting from the $(n+1)$ vertices of the right isosceles simplex, shrinking the lengths between adjacent points and copying the resulting stellar figure in each of the coordinate directions. It is stellar in the sense that it looks like a star or asterisk. The construction of shrinking and duplicating is completely analogous to the construction of Sierpinski's triangle except in our description, we evacuated the central hunk of the n-simplex. But from the perspective of the analysis of dimension, and the general similarity of form, we see that the image of the binary Pascal triangle under the tent map is contained entirely within the Sierpinski n-simplex. From the visually available examples in dimension 2 and 3, we see that the Pascal figure gives a good approximation of the Sierpinski figure.

The general similarity between the binary Pascal n-simplex and the Sierpinski n-simplex, is understood via two principals. The first part of the Parity Theorem gives that the points colored black land on the Sierpinski set. The second part gives that the removed central portions of the n-simplex are contained within the white regions in the image of the binary Pascal n-simplex under the tent map.

6.5. Conclusion. Wow! Starting in an innocent fashion from considering a line segment in the plane and its generalizations: triangle, tetrahedron, and beyond, we discussed a plethora of ideas. Specifically, we described the chaos game in the n-simplex, showed how this approximates the incredibly intricate Sierpinski n-simplex. This Sierpinski figure generalizes the Sierpinski triangles and imitates Cantor's construction. It is an uncountable set of points each of which has a neighborhood that contains points of the set. Yet this set has no interior. It is like the glutinous matrix of a stale frosted orange drink. We demonstrated how to rotate and project these higher dimensional sets to the coordinate plane and we will use these projections to construct our movies.

In a seemingly unrelated topic, we examined the multinomial coefficients and we showed how these can be understood in terms of a recursion relation that generalizes the famous Pascal relation that you should have learned in school. Then by a simple principal of black and white coloring, we demonstrated a similarity of form between the multinomial coefficients and the Sierpinski Figure. The meaning of this similarity is that either the binary Pascal figure or the chaos game can be used to illustrate the Sierpinski sets. Specifically, for any point in the Sierpinski set and for any neighborhood, the chaos figure or the binary Pascal figure (after the tent map is applied) are within the neighborhood of the given point. Alternatively, in the real world, the world in which points are plotted with thickness, the plotted points from these countable figures simulate the full set of points in the Sierpinski set.

Much of this mathematics is calculus related, but it is not specifically calculus dependent. In fact, it is the type of thing that we would hope a well rounded student with a mathematics related undergraduate degree would be able to understand without breaking a sweat. We hope that you agree that the discussion gives a manifestation of beauty from a realm in which you heretofore may have only glimpsed.

The next chapter consists of the computer code that I used to construct the movies on the associated disk. Please take the code, modify, and improve it. Make it your own, and when you do thank me for the initial inspiration. My commentary in the next chapter is contained entirely within the computer comment lines.

Chapter 3

Computer Code

The rule of computer code should be (1) I have figured out how to solve *this* problem. You are free to use my code to address *this* problem, but please acknowledge my contributions. (2) You may modify my code to solve *that* problem, but if you use my code, then you should make your code readily available to others who may want to solve yet another problem. Of course, I am articulating an overly idealistic point of view. Various shareware, GNU software, and copy-left material follow these rules. Pure mathematicians, as a general rule, follow these rules. But the commercial applications of pure mathematics are usually decades away from their introduction. I have been told that cellular telephones need to take into consideration calculations in general relativity to adjust clock speed. This use does not net the Einstein estate any financial rewards. Computer software is among the most commercial of enterprizes. The software here certainly can be used for commercial endeavors. In particular, the technique of rotation and projection can be applied to any multi-variate data set. I should care, but I don't.

My rewards for doing mathematics are (1) the joy I get in doing it. (2) I am gainfully employed and if the university, does not support my intellectual endeavors, it, at least, tolerates them. (3) I get a modicum of respect from a number of colleagues, most of who I consider more clever than I. So if you use this code, adapt it, make it more efficient, and modify it more power to you. Please, acknowledge my contribution. It will give me a cheap thrill.

If you make this a commercial endeavor and make lots of money from it, throw a little my way. That will make my family happy and it will keep the money from going to the lawyer who would handle the suit.

1. Program: Making files

```
(* PROGRAM MAKING FILES *)
(* Note : This line is a comment. It is a comment since
it is set off by a left parenthesis followed by an
asterisk and closed by an asterisk followed by a
right parenthesis.

Here is the first program that I used to make the
graphical images. Its  purpose is to define a set
of matrices, to define a set of some 70,000 points
in higher dimensional space, rotate the collection of
these points and project them. The resulting set of
points are stored in files which the next program
opens.

Mathematica programs are grouped into cells. The cells
are like paragraphs: they function to define a certain
idea. The collection of  these ideas is the
program. *)


(*first cell *)
vera[prop_] := If[TrueQ[prop], 1, 0];
(*vera stands for veracity. This is a type of
characteristic function. If a proposition is true,
vera returns the value 1 otherwise it returns 0.
I have since learned of a mathematic command ‘‘Boole"
that achieves the same thing. *)
```

```
val[t_, a_, b_] := vera[a <= t]  vera[t < b] ;
vem[t_, a_, b_] := vera[a <= t]  vera[t <= b] ;
(* val defines an interval.
val is the characteristic function of the interval [a,b)
which includes a but does not include b.
vem does the same for the closed interval [a, b].
I don't remember why I called it vem. *)

lyle[n_, j_, t_] := Floor[n/2] t - j + 1;
(* lyle rhymes with tile which is found on the floor.
Tiles lies in a line an this is essentially a linear
function. Since the n-simplex is symmetric I only
have to rotate trough roughly n/2 different angles.
I used the floor function to get the ``roughly" part
right. Floor is the greatest integer less than or equal
to the given number. *)

booze[n_] :=
 Flatten[Table[{i, j}, {i, 1, n - 1}, {j, i + 1, n}], 1];
(* booze is the set of all two element subsets of 1
through n. Choose is the word booze was meant to
rhyme with.  The mathematica function ``Table"
creates a table, but this table has extra brackets
in it. Flatten[*,1] removes some extra {}s.*)

wine[j_, n_, t_] := ReplacePart[
      ReplacePart[
        ReplacePart[
          ReplacePart[IdentityMatrix[n],
            Cos[ 2Pi lyle[n, j, t] ],
            {booze[n][[j]][[1]],
              booze[n][[j]][[1]]}],
          Cos[2Pi lyle[n, j, t]], {booze[n][[j]][[2]],
            booze[n][[j]][[2]]}],
          -Sin[2Pi lyle[n, j, t]], {booze[n][[j]][[2]],
             booze[n][[j]][[1]]}],
```

```
      Sin[ 2Pi lyle[n, j, t]], {booze[n][[j]][[1]],
      booze[n][[j]][[2]]}];
(* Why wine? booze-wine it is becoming thematic.
wine is a rotation matrix. This is the matrix
M_{ij}(theta) that rotates in any pair of coordinates.
Except for the $(i,i),(i,j),(j,i) and (j,j)$ positions,
the matrix is an identity matrix.
``Replace Part[list,element,place]" replaces the
'place' entry of 'list' with 'element.' It is applied
4 times because there are 4 entries to  be changed.
The trig functions are evaluated at linear expressions
to make them affect full rotations at the appropriate
time.*)


tire[t_, n_] :=
    Sum[val[t, (j - 1)/Floor[n/2], j/Floor[n/2]]
        wine[j, n, t], {j, 1, Floor[(n - 2)/2]}]
    + vem[t, (Floor[n/2] - 1)/Floor[n/2], 1]
    wine[Floor[n/2], n, t];

(* you rotates tires, right? The function tire takes
care of the successive rotations. This is the matrix
that first rotates in the (12) plane, then in the (13)
plane and so forth. It runs on characteristic
  functions times the appropriate values of wine.*)

(*This next line clears the terms stuff, pts, pigment,
coloredpts2plot and xset, for use below.

stuff is the table of random numbers.
pigment is the color that I want to color them.
colored pts2plot are the  points with certain
colors associated to  them.
xset is the set of rows of the identity matrix.*)
Clear [stuff];
Clear[pigment];
```

```
Clear[coloredpts2plot];
Clear[xset];


xset[n_] := IdentityMatrix[n];


(* zillions is the number of points I am going to graph.
zillions should be in principal a very large number.
70,000  is pretty good ... *)

zillions = 70000

(* This is the end of the 1st cell. except for the last
bit, it is pretty self-contained. It defines the
matrices of rotations, clears some variables, and
defines xset and zillions.
zillions can be assigned to suit the user's  pleasure. *)



(* second cell *)
(*This line generates a table of integers which I
will use to graph my figure. The number of integers
I will use is called n.
This number is actually n + 1 in the text,
but it is a pain to change that now. *)
n = 5;
(* Here n is one more than the dimension of the n
simplex being graphed.  change it at will. *)
stuff = Table[Random[Integer, {1, n}], {zillions}];
(* Ok, so stuff is a bunch of  stuff. It is a set of
70,000 random numbers where each is taken to be
between 1 and n. 'stuff' corresponds to the list
A in the text.*)
```

```
(* s is the distance that  my points moves.
Mostly s =0.5. but one graph used a different
value of s. This is a parameter of the chaos game.
When you play with the program, change s and see
what happens.*)
s = .5
(* end of second cell *)

(* third cell *)

eh[m_] := {Table[Re[E^(2 Pi j I/m)], {j, 0, m - 1}],
    Table[Im[E^(2 Pi j I/m)], {j, 0, m - 1}]}

(* Canadian joke, eh? Somewhere else there is a variable
a, or I copied this from another program.
This is the (2 x m) matrix that projects a simplex
onto a regular polygon. The argument of  the
function is m. Later this will  be the
n+1 of the text or n of the program. Here the
complex valued exponential function is being
used. e^( i theta)= cos(theta)+ i sin(theta).
This matrix is A_m from the text. *)

Table[Dot[eh[n], tire[t, n]],
{t, 0, 1, 0.0025}] >> mat5.dat;
(* This line pumps the rotations together with
the projections into a file. Strictly speaking,
we only need to execute this  portion of the
program once, save the file and never worry about
it again. The value 0.0025 gives 401 stills to the
movies, the variable t is the time parameter of
the movie. The function 'Dot' computes the
matrix product. Here this product is
completely independent of anything else
in the program. That is once you have this,
you can used these matrices for any data set.*)
(* end of 3rd cell *)
```

```
(*4th cell)

Clear[added]
(* one gets in the habit of clearing variables
every once in a which *)
(* end of 4th cell *)


(*5th cell *)
(* Starting at the vertex j (which is randomly
chosen) I move halfway towards the next random
vertex in the sequence for the entire sequence.
My function "added" is defined recursively. In
other words,the next value of the function is
determined by the previous values of the
function. The line added[j_] : (added[j]
stores values. N[** **] gives a numeric
approximation of this point. An approximation
must be used to avoid a Mathematica error.

To sum this up, this line of code is what
actually generates the coordinates of my
entire figure. *)

added[j_] := (added[j] =
Release[
 N[ s added[j - 1] + (1 - s) xset[Floor[n]]
            [[stuff[[j]]]]]]);
        added[1] = xset[n][[stuff[[1]]]]
(* This last bit initializes the sequence
at some vertex of the figure *)
(* end of 5th cell *)

(* 6th cell *)
Table[added[k], {k, 1, zillions}]
>> rand5.dat
```

```
(*This line is crucial to get right.
You pump the data into an external file.
To be safe change the name of the file
into which you send the data.*)
(* end of 6th cell *)


(* 7th cell *)
Clear[pigment];

(* This line tells my computer what color
to make the points it computed.
This color is determined by what vertex
comes next in my pattern and then the
point it creates is labeled a color
accordingly. *)
pigment[m_] :=
    Thread[
      Hue[Table[
          N[Floor[
          N[( stuff[[j]])/m, 2]*100]/100, 2],
          {j, 1, zillions}]]];
(* As a matter of fact this line took a
lot of programming time to get right. *)
pigment[n] >> pig5.dat
(* The file  name should reflect the name
above. Again, I am pumping data into a file
to call it up later *)

(* end of 7th cell *)



(*Run the program as is. Then shut down
mathematica and open the next program called
"Drawing Files." *)
```

2. Drawing Files

```
(* 1st cell *)
(* This is the program "Drawing Files."
It brings in the data from the program
"Making Files," and processes the data
into graphical format. I found that I
needed these separate programs on the
laptop on which the images were created.
Its random access memory is a bit small:
1/2 gig. More  modern machines may do
better. So I have to write half of the
graphics at a time. I will  explain how
to do this later. This program,
creates the movies of rotation of the
Fractal (n-1)-simplex. These can be
displayed within the program,
of mapped to files to be printed.
After they are mapped to files,
we can process them with another program.
This first line brings the data into
the program. *)

piggy = Get["pig5.dat"];
manure = Get["mat5.dat"];
ale = Get["rand5.dat"];

(* piggy came from pigment.
Manure often comes from pigs as well as other
animals. Within the program, matrix
malaproped into mature. The variable
manure was far enough away so that
the Mathematica spelling police
would not mind, but close enough so
that I could remember what was what.
The variable ale was a morph of all;
namely all the data within the simplex. *)
```

```
(* end 1st cell *)

(* 2nd cell *)

Length[manure] (* This line is not
necessary, but I keep  it  to see that
manure has been loaded into memory.
A colleague once told me that he suffered
from CRT --- can't remember s ***.
Another comments on the amount of useless s ***
that I remember. *)
(* end 2nd cell *)


(* 3rd cell *)
Length[piggy] (* ditto. Just how many points
to be plotted. *)

(* end 3rd cell *)


(* 4th cell *)
fish2fry[t_] :=
  Table[{piggy[[j]], Point[manure[[t]].ale[[j]]]},
  {j, 1,Length[piggy]}]
(* Ok, this variable was going to be points2plot,
but Point and Plot are mathematica commands.
I  must have needed an action statement.
Also in another program I am writing,
fish names are used for variables.
We seldom eat fried fish in our house ---
strange for people living in the south.
Fried catfish is pretty good around here. *)
(* end 4th cell *)

(* Now the instructions to the used become
complicated. I like detailed movies.
```

```
But with 70,000 points they get
sort of memory intensive. So I execute
the program several times for different
data ranges. *)

(* 5th cell *)

Table[Show[Graphics[{AbsolutePointSize[1],
                     fish2fry[t]}],
PlotLabel -> StyleForm[t, "Section",
FontColor -> GrayLevel[1]],
AspectRatio -> 1, Axes -> False,
Background -> GrayLevel[0.03],
ImageSize -> {(72*6), (72*6)}],
{t, 1, Floor[Length[manure]/4]}]
(* With more memory, you could avoid
cutting this into quarters.
Once you get the output from here,
quit the kernel, cut this output,
paste it into another notebook,
and save and close that notebook.
Shut down the computer or find a way
to  clear the buffer's memory.
Otherwise the paste may  crash.
Then don't run this cell,
  but run the next. *)
(* end 5th cell *)


(* 6th cell *)
(* Restart the program, and run everything
but the previous cell. Instead run this cell.
Then quit  the kernel. Cut the output and
paste it  into the notebook, at the end where
you saved the output of  the preceding cell *)
Table[Show[Graphics[{AbsolutePointSize[1],
fish2fry[t]}],
```

```
PlotLabel -> StyleForm[t, "Section",
FontColor -> GrayLevel[1]],
AspectRatio -> 1, Axes -> False,
Background -> GrayLevel[0.03],
ImageSize -> {(72*6), (72*6)}],
{t, Floor[Length[manure]/4] + 1,
Floor[2Length[manure]/4]}]
(*Quit the kernel. Cut the output  of
this cell, and paste it into the notebook
that contains the previous output.
Save that notebook, and close it.
Shut down the computer or find a
way to  clear the buffer's memory.
Otherwise the paste may  crash.
Then restart the program,
but don't run the previous cell or
this cell. Run up  to that that point,
then run the next cell*)
(* end 6th cell *)



(* 7th cell *)

(* Run this cell. Then quit the kernel.
Cut the output and paste it into
the notebook, at the end where you
saved the output of  the preceding cell *)
Table[Show[Graphics[{AbsolutePointSize[1],
fish2fry[t]}],
    PlotLabel -> StyleForm[t, "Section",
    FontColor -> GrayLevel[1]],
    AspectRatio -> 1, Axes -> False,
    Background -> GrayLevel[0.03],
    ImageSize -> {(72*6), (72*6)}],
    {t, Floor[2 Length[manure]/4] + 1,
    Floor[3Length[manure]/4]}]
```

```
(*Quit the kernel. Cut the output
 of this cell, and paste it into
 the notebook that contains the
 previous output. Save that notebook,
 and close it. Shut down the computer
 or find a way to  clear the buffer's
 memory. Otherwise the paste may crash.
 Then restart the program, but don't
 run the previous cell or this cell.
 Run up  to that that point,
then run the next cell*)
(* end 7th cell *)


(* 8th cell *)

(* Run this cell. Then quit  the kernel.
      Cut the output and paste it  into the notebook,
  at the end where you saved the output of  the preceding cell *)
Table[Show[Graphics[{AbsolutePointSize[1],
fish2fry[t]}], PlotLabel ->
   StyleForm[t, "Section",
   FontColor -> GrayLevel[1]],
   AspectRatio -> 1,
   Axes -> False,
   Background -> GrayLevel[0.03],
   ImageSize -> {(72*6), (72*6)}],
   {t, Floor[3 Length[manure]/4] + 1,
    Length[manure]}]
(* end 8th cell *)


(* 9th cell *)
(* for your enjoyment *)
mole = Show[Graphics[{AbsolutePointSize[1],
            fish2fry[1]}], AspectRatio -> 1,
            Axes -> False, Background -> GrayLevel[1],
```

```
            ImageSize -> {(72*6), (72*6)}]

(* Here is a method of developing the first
(or any) picture of the movie.
By adjusting the argument of fish2fry,
you can see any particular still.
To  do so may be helpful if you want
to include more points.
Notice that the background color is
set to white here. *)

(* end 9th cell *)


(* 10th cell *)

Export["initial.eps", mole, "EPS"]
(* This line exports the single still  to a file
names initial.eps. Such a file  can be used by an
auxiliary program such as LaTeX for a graphics object.*)
(* end 10th cell  *)


(* 11th cell *)
Do[narf = StringJoin["foursimp", ToString[ t ], ".eps"];
  Export[narf,
   Graphics[(*{, ,}*) {AbsolutePointSize[1], fish2fry[t]},
   AspectRatio -> 1,
   Axes -> False, Background -> GrayLevel[0.03],
   ImageSize -> {(72*4), (72*4)},
   PlotLabel ->
   StyleForm[{t, "The Four simplex"}, FontColor -> GrayLevel[1]]],
    "eps"(*, ImageOffset -> {200, 300}*)], {t, 1, Length[manure]}]

(* This line exports the full set of stills to a
sequence of eps files. These can then be read by
an external program such as TeX and entered into
```

```
the book form. This line is literally what I used
to make the marginal illustrations. However, I
did not set a path in Mathematica to a file
that can be read by TeX. So I had to  cut
and paste those into  TeX's directory by hand.
It seemed easier that way. I  don't know why*)
```

3. Unfolding files

```
(* Note : This program is a virtual repeat of the
file "making files." It is concerned with making
the images that unfold from a Sierpinski triangle
to a Sierpinski 5 simplex. Minor modifications
can be made to make this unfold more.
I will  point you  to the line at which
changes can be made. Other comments
that appeared above will be suppressed
including a description of  cell
structure.*)

Clear [stuff];
Clear[pigment];
Clear[coloredpts2plot];
Clear[xset];


xset[n_] := IdentityMatrix[n];


(* zillions is the number of points I am
going to graph. 40,000 is pretty good.
I used fewer points here because
during the unfolding a lot of these points
are overlapping. *)

zillions = 40000;
```

```
n = 6;
(*ATTENTION IF YOU  WANT TO UNFOLD A k - simplex,
then set n = k + 1. To unfold a 4 simplex use  n = 5.*)
stuff = Table[Random[Integer, {1, n}], {zillions}];
s = .5;
Clear[added];
added[j_] := (added[j] =
  Release[N[ s added[j - 1]
     + (1 - s) xset[Floor[n]][[stuff[[j]]]]]]);
     added[1] = xset[n][[stuff[[1]]]];
Table[added[k],
     {k, 1, zillions}] >> rand6fld.dat
(*This line is crucial to get right.
You pump the data into a file,
but you should rename that set on each run.
To be safe change the name of the file
into which you send the data.
Here we pump  the data into a file
called rand6fld.
We will use these data later.*)

Clear[pigment];
pigment[m_] :=
    Thread[
      Hue[Table[
          N[Floor[N[( stuff[[j]])/m, 2]*100]/100, 2],
          {j, 1, zillions}]]];


pigment[n] >> pig6fld.dat
(* The file name should reflect the name
above. Again, I am pumping data into a
file  to call it up later *)

  Clear[pendulus]; Clear[flick];
```

```
  Clear[booger]; Clear[eye]; Clear[nose]
vera[prop_] := If[TrueQ[prop], 1, 0];

val[t_, a_, b_] := vera[a <= t]  vera[t < b] ;
vem[t_, a_, b_] := vera[a <= t]  vera[t <= b] ;

afro[x_, y_, u_, t_] := (u + 1 - t)x + (t - u)y;
pendulus[u_, t_, n_] :=
 Table[afro[E^(2 Pi I  (u + 2)/(u + 3)),
  E^(2 Pi I (u + 2)/(u + 3)),
  u, t], {j, 1, n - u - 2}];
flick[u_, t_, n_] :=
    Table[afro[E^(2 Pi I   k/(u + 3)),
    E^(2 Pi I  k/(u + 4)), u, t], {k, 0,
        u + 2}];
booger[u_, t_, n_] :=
Join[flick[u, t, n],
pendulus[u, t, n]];
nose[u_, t_, n_] := {Re[booger[u, t, n]],
Im[booger[u, t, n]]};
eye[t_, n_] :=
  Sum[val[t, u, u + 1]nose[u, t, n],
  {u, 0, n - 3}] + vem[t, n - 3, n - 2]nose[n - 3, t, n];
(* Remember from the file Making Files,
that vera is a truth function that
depends on the truth value of a
proposition. The functions val and
vem are  characteristic functions of intervals;
val characterizes an interval closed on the
left and open on the right,
vem is closed on both sides.
The function "afro" is a line (lions are from Africa?)
that gives x when t =u and y when t =
u + 1. The function pendulus is that
which is appended to an array so that
it matches the dimension of the simplex
that is unfolding. The function "flick,"
```

```
flicks one vertex off of the other,
"booger" just arranges flick and
pendulus into one array.
The function "nose" makes this a genuine 2 x n array.
The function "eye" sums up the unfoldings. *)


Table[eye[t, 5], {t, 0, 3, 0.0075}]
    >> eye6fld.dat
(* This maps a table of (2 x (n + 1))
matrices into a file. *)

(* Before executing this line,
QUIT the Kernel of Mathematica. The data
have been mapped to three files.
Now we load those into these files
and generate the graphs.*)
piggy = Get["pig6fld.dat"];
manure = Get["eye6fld.dat"];
ale = Get["rand6fld.dat"];
(* This line brings the data back into the
program so that it can generate the figures. *)

(*Next execute this line to feed the
data into a useful form. *)
fish2fry[t_] :=
Table[{piggy[[j]], Point[manure[[t]].ale[[j]]]},
      {j, 1, Length[piggy]}];

(* Now execute this line, quit the kernel,
cut the graphics and paste them into a
notebook. Save that notebook! After
doing so, go up two cells start again,
and run those cells above and the cell below. *)

Table[Show[Graphics[{AbsolutePointSize[1],
    fish2fry[t]}],
```

```
    PlotLabel -> StyleForm[{t, "unfolding"},
    "Section", FontColor -> GrayLevel[1]],
    AspectRatio -> 1, Axes -> False,
    Background -> GrayLevel[0.03],
    ImageSize -> {(72*6), (72*6)}], {t, 1, 100}]

(* In step 2 execute this line, quit the
kernel, cut the graphics and paste them at the
end of the bottom of the notebook you made
in the first step.  Save that notebook! After
doing so, go up three cells start again,
and run the first 2 of those cells above and
the cell below. *)

Table[Show[Graphics[{AbsolutePointSize[1], fish2fry[t]}],
 PlotLabel ->
 StyleForm[{t, "unfolding"}, "Section",
   FontColor -> GrayLevel[1]],
   AspectRatio -> 1, Axes -> False,
   Background -> GrayLevel[0.03],
   ImageSize -> {(72*6), (72*6)}], {t, 101, 200}]

(* In step 3 execute this line, quit
the kernel, cut the graphics and paste
them into a separate notebook.
Save that notebook. After doing
so, go up four cells start again,
and run the first 2 of those cells
above and the cell below. *)


Table[Show[Graphics[{AbsolutePointSize[1],
      fish2fry[t]}], PlotLabel ->
      StyleForm[{t, "unfolding"},
      "Section", FontColor -> GrayLevel[1]],
      AspectRatio -> 1, Axes -> False,
      Background -> GrayLevel[0.03],
```

```
      ImageSize -> {(72*6), (72*6)}],
      {t, 201, 300}]

(* In step 4 execute this line,
quit the kernel, cut the graphics
and paste them into a separate notebook.
Save that notebook. After doing so,
go up five cells start again,
and run the first 2 of those cells
above and the cell below. *)

Table[Show[Graphics[{AbsolutePointSize[1],
 fish2fry[t]}], PlotLabel ->
 StyleForm[{t, "unfolding"},
 "Section", FontColor -> GrayLevel[1]],
 AspectRatio -> 1, Axes -> False,
 Background -> GrayLevel[0.03],
 ImageSize -> {(72*6), (72*6)}],
 {t, 301, 401}]


Do[narf = StringJoin["unfold", ToString[ t ], ".eps"];
  Export[narf,
    Graphics[(*{, ,}*) {AbsolutePointSize[1],
    fish2fry[t]}, AspectRatio -> 1,
      Axes -> False, Background -> GrayLevel[0.03],
      ImageSize -> {(72*6), (72*6)},
      PlotLabel -> StyleForm[{t, "Unfolding"},
      FontColor -> GrayLevel[1]]],
    "eps"(*, ImageOffset -> {200, 300}*)],
    {t, 1, Length[manure]}]

(* This line exports the full set
of stills to a sequence of eps files.
These can then be read by an external
program such as TeX and entered \
into the book form.
```

```
This line is literally what I used to make
the page illustrations.
However, I did not set a path
in Mathematica to a file that can be read by TeX.
So I had to  cut and paste those
into  TeX's directory by hand.
It seemed easier that way.*)
```

Chapter 4

Glossary of Terms

In this chapter, I made an attempt to give a glossary of all the terms that I have used herein. The organization of the glossary entries are first alphabetical Greek letters that were used, then the Roman letters in alphabetical order, and finally the corresponding terms in alphabetical order. The glossary, then, is highly redundant.When feasible, I gave more than one definition, and sometimes I pointed to other terminologies that are standard within the field. These are some of the terms with which an educated person should be familiar. If you are unfamiliar when you finish this book, you didn't read carefully enough.

- Δx —- the change in x. If (x_0, y_0) and (x_1, y_1) are points in the plane, then $\Delta x = x_1 - x_0$. Δy is defined similarly.
- ϵ-neighborhood — an ϵ neighborhood of a point $\vec{x}$ consists of those points $\vec{y}$ such that the distance between $\vec{x}$ and $\vec{y}$ is strictly smaller than ϵ. **i.e.**

  $$N_\epsilon(\vec{x}) = \{\vec{y} : \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2} < \epsilon.$$

  In principal ϵ is chosen to be an arbitrarily small positive real numbers.

- ζ_{n+1} — the primitive $(n+1)$st root of 1 in the complex plane. The complex number ζ_{n+1} is chosen to satisfy $(\zeta_{n+1})^{n+1} = 1$ and have the smallest positive angle (as measured from the x-axis) among those that do. This is the point on the unit circle subtended by the angle $\frac{2\pi}{n+1} = 360°/(n+1)$. In complex coordinates, $\zeta_{n+1} = \cos\frac{2\pi}{n+1} + i \sin\frac{2\pi}{n+1}$ where $i^2 + 1 = 0$; that is i is a square root of -1.
- θ — typically the symbol used to denote an angle. Angle measurements are given in **radians**. The radian measure of an angle is the length of the arc along the unit circle that is subtended by the angle.
- π — the Greek letter "pi" used to denote the area of a circle of radius 1 or equivalently, half the circumference of a circle of the same radius.
- χ_Y — the characteristic function defined on a set X that indicates a set Y. For example,

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

- $[0,1]$—the closed interval from 0 to 1. $[0,1] = \{t \in R : 0 \leq t \leq 1\}$. More generally, $[a,b] = \{t \in R : a \leq t \leq b\}$
- A — The letter A is used in different contexts herein. (1) The sequence A is an original n-ary sequence from which the chaos game is played. In the sequence $A = (a_1, a_2, \ldots)$ each a_j is chosen from among the elements of $\{1, 2, \ldots, n+1\}$. (2) $A = (a_{ij})$ — an $(n \times n)$ matrix. This is a square array of numbers $a_{ij} \in \mathbb{R}$ for each $i, j = 1, 2, \ldots, n$. A matrix can be thought of as a transformation to space, or as a transformation of an underlying piece of space. (3) The first factor in the matrix product $A \cdot \vec{x}$. The product of the matrix A and the vector $\vec{x}$ is the vector $\vec{y}$ whose ith component is $y_i = \sum_{j=1}^{n} a_{ij} x_j = a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n$. (4) The matrix A_m is the projection of m space onto the plane in such a way that the standard coordinate vectors go to the vertices of a regular polygon.

- $B_{n+1} = \{e_1, e_2, \ldots, e_{n+1}\}$ — this is the set of standard basis vectors in $\mathbb{R}^{n+1}$.
- $C_n^{i,j}$ — the cardinal (hyper)-plane in n-dimensional space that is given by the equations $x_i = 0$ and $x_j = 0$. For example $C_3^{1,2} = \{(0, 0, z) \in \mathbb{R}^3 : z \in R\}$.
- $\cos\theta$ —The cosine of an angle θ is the x-coordinate of the point on the unit circle that is subtended by the angle θ.
- e — the base of the natural logarithm, not to be confused with the standard coordinate basis defined below. The number e is a real number, and it can be computed as the limit $e = \lim_{n\to\infty}(1 + \frac{1}{n})^n$.
- e_j — the jth standard coordinate vector that consists of a (possibly empty) sequence of 0s followed by a single 1 in the jth position and another (possibly empty) subsequence sequence of 0s. More specifically, $e_j = (\underbrace{0, \ldots, 0}_{j-1}, 1, \underbrace{0, \ldots, 0}_{n+1-j})$
- I_k — the $(k \times k)$ identity matrix. All of its diagonal entries are 1s and all non-diagonal entries are 0. This matrix fixes all of the points in k-dimensional space.
- $$\binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \frac{(k_1 + k_2 + \cdots + k_n)!}{k_1!k_2! \cdots k_n!}.$$

  In this notation, the binomial coefficient is given as
  $$\binom{r}{k} = \binom{r}{k, r-k}.$$

  The expression $[k_1, k_2, \ldots, k_n]$ is an internal notation for the multinomial coefficient:
  $$[k_1, k_2, \ldots, k_n] = \binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n}.$$
- $\lim_{m\to\infty} f(n)$ — the limit of a real valued sequence of numbers. This limit exists and is equal to a number L if and only if for every $\epsilon > 0$ there is an integer $N > 0$ such that $|f(n) - L| < \epsilon$ whenever $n > N$.

- $\log_2(y) = x$ — the base two logarithm. The equation means that $2^x = y$. It only makes sense when y is a positive quantity. The change of base formula gives $log_2(y) = \ln(y)/\ln(2)$ where ln indicates the natural logarithm — the log to the base e.
- $M_{ij}(\theta)$ — this is a rotation of n-dimensional space through an angle θ that fixes the cardinal hyper-plane $C_n^{i,j}$. It is given in matrix form as

$$M_{ij}(\theta) = \begin{array}{r} \\ i\text{th} \rightarrow \\ \\ j\text{th} \rightarrow \\ \\ \end{array} \begin{bmatrix} I_{i-1} & 0 & 0 & 0 & 0 \\ 0 & \cos\theta & 0 & \sin\theta & 0 \\ 0 & 0 & I_{j-i-1} & 0 & 0 \\ 0 & -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 0 & I_{n-j} \end{bmatrix}.$$

- $\mathbb{N} = \{1, 2, 3, \ldots\}$ — the natural numbers. A key property of the natural numbers is that ever number has a successor. This is the basis for mathematical induction.
- N usually a large natural number or a specific natural number.
- n-ary — for example binary or ternary. The n-ary expression of a number is to write it as a sum of multiples of powers of n. The multipliers are taken from 0 through $n-1$. The most familiar expression is the decimal expression. For example, the number commonly expressed as nine thousand, four hundred, and eighty three is written as $9,483$ to express it as $9 \times 10^3 + 4 \times 10^3 + 8 \times 10^1 + 3 \times 10^0$. In n-ary notation the same number can be expressed differently. The decimal number 4 in binary notation is expressed as 100_2.
- n-simplex — the convex hull of the the unit coordinate vectors in $(n+1)$-space. The n-simplex is denoted by S_n and it is given as $S_n = \{(x_1, x_2, \ldots, x_{n+1}) \in R^{n+1} : \sum_{j=1}^{n+1} x_j = 1 \ \ \& x_j \geq 0 \ \text{ for all } \ j = 1, 2, \ldots n+1\}$
- $n!$— n factorial. The quantity $0!$ is by definition 1. If $n > 0$, then $n! = n(n-1)!$. For example, $3! = 3 \cdot 2 \cdot 1$. The quantity

$n!$ counts the number of ways that a set with n distinct elements can be arranged.

- $\binom{n+1}{2} = \frac{(n+1)n}{2}$ — the symbol is read $n+1$ choose 2; it is the number of ways of choosing an un-ordered pair from a set with $n+1$ elements.
- $\mathbb{Q}$ denotes the set of rational numbers. These are the set of fractions a/b where a and b are integers, and $b \neq 0$. Usually, a and b are chosen so that they have no common factors. More precisely, we say that $a/b = c/d$ if and only if $ad = bc$. Here neither b nor d is 0.
- $\mathbb{R}$ — the set of real numbers. The real number are the numbers of infinite precision. Any real number can be expressed in decimal notation as a sign ($\pm$), and an infinite string of digits taken from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. The string starts on the left, has a decimal to indicate place value and continues infinitely far to the right. We can write, $X = x_n x_{n-1} \ldots x_1 x_0 . x_{-1} x_{-2} \ldots$ where $x_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ to mean
$$X = \sum_{k=-n}^{\infty} x_{-k} 10^{-k}.$$
- $\mathbb{R}^2$— the coordinate plane. The coordinate plane is the set of ordered pairs
$$\mathbb{R}^2 = \{(x, y) : x, y \in \mathbb{R}\}$$
distance between a pair of points $\vec{v} = (x_1, y_1)$ and $\vec{w} = (x_0, y_0)$ in the plane is give by the expression
$$d(\vec{v}, \vec{w}) = \sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{(x_1 - x_0)^2 + (y_1 - y_0)^2}$$
- $\mathbb{R}^n$ — n-dimensional space or simply n-space. This is the set of ordered n-tuples
$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R} \text{ for all } j = 1, 2, \ldots, n\}.$$
Distance between a pair of points $\vec{v} = (x_1, x_2, \ldots, x_n)$ and $\vec{w} = (y_1, y_2, \ldots, y_n)$ in n-space is given by
$$d(\vec{v}, \vec{w}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

to avoid doubling subscripted variables, the notation changed from the previous glossary entry.

- $\sin\theta$ — the sine of an angle θ is the y-coordinate of the point on the unit circle that is subtended by the angle θ.
- $\lfloor X \rfloor$ — the greatest integer less than or equal to X.
- $\in$ — **in** or is an element of. To say that $x \in S$ (x is in S) is to say that x is an element of the set S.
- **Albert** — the oldest of my three sons. His studies instigated this volume.
- The **Area** of a circle of radius 1 is the number $\pi \approx 3.14159$. This is an irrational number so its decimal expansion continues forever without repeating.
- **binary or ternary** — n-ary — the n-ary expression of a number is to write it as a sum of multiples of powers of n. The multipliers are taken from 0 through $n-1$. The most familiar expression is the decimal expression. For example, the number commonly expressed as nine thousand, four hundred and eighty three is written as $9,483$ to express it as $9 \times 10^3 + 4 \times 10^3 + 8 \times 10^1 + 3 \times 10^0$. In n-ary notation the same number can be expressed differently. The decimal number 4 in binary notation is expressed as 100_2.
- The **binomial theorem** gives an explicit closed form calculation of the coefficient of $x^k y^{n-k}$ in the expansion of $(x+y)^r$:

$$(x+y)^r = \sum_{k}^{r} \binom{r}{k} x^k y^{r-k}.$$

- The **cardinal (hyper)-plane in n-dimensional space** is denoted as $C_n^{i,j}$ and is given by the equations $x_i = 0$ and $x_j = 0$ in n-space. For example $C_3^{1,2} = \{(0,0,z) \in \mathbb{R}^3 : z \in R\}$.
- A **Characteristic function** is a function, χ_Y that is defined on a set X that takes values 1 precisely on elements of a subset Y and takes the value 0 elsewhere on X. We use the characteristic function defined on the real numbers that

indicates closed intervals

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } \ a \leq x \leq b \\ 0 & \quad \text{otherwise} \end{cases}$$

- **choose** — $\binom{n+1}{2} = \frac{(n+1)n}{2}$ — the symbol is read $n+1$ choose 2; it is the number of ways of choosing an un-ordered pair from a set with $n+1$ elements. In general, $\binom{a}{b} = \frac{(a)!}{b!(a-b)!}$ — binomial coefficient. The symbol is read $n+1$ choose k; it is the number of ways of choosing an un-ordered k-tuple of points from a set that has $n+1$-elements.
- **closed interval** — an interval of real numbers in which the end points are included. When a and b are real numbers the closed interval $[a, b] = \{t \in R : a \leq t \leq b\}$.
- **Convex set** — a set is convex if for any pair of points in the set the straight line that joins these points is within the set.
- **Convex Hull of a set**— the intersection of all convex sets that contain the given set.
- The **cosine of an angle** θ, denoted $cos\theta$ is the x-coordinate of the point on the unit circle that is subtended by the angle θ.
- **Coordinate space** also called, n-dimensional space, n-space, or $\mathbb{R}^n$. This is the set of ordered n-tuples

$$\mathbb{R}^n = \{(x_1, x_2, \ldots, x_n) : x_j \in \mathbb{R} \ \text{ for all } \ j = 1, 2, \ldots, n\}.$$

Distance between a pair of points $\vec{v} = (x_1, x_2, \ldots, x_n)$ and $\vec{w} = (y_1, y_2, \ldots, y_n)$ in n-space is given by

$$d(\vec{v}, \vec{w}) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots + (x_n - y_n)^2}.$$

The **coordinate plane** is the case in which $n = 2$.

- **Eleanor** my mother's name used here to denote the set

$$\{(x_1, x_2, \ldots, x_n) : \sum_{i=1}^{n} x_i \leq 1, \quad \& \quad x_i \geq 0 \ \text{ for } i = 1, 2 \ldots, n\}.$$

This is a right isosceles n-simplex of volume $1/n!$. It is right because the coordinate facets meet each other at right angles. It is isosceles because the $(n-1)$-simplicial facet have edges that are of equal length. The name came to mind because it was a particularly beautiful model (as so was my mother), and a drawing of her ring appears elsewhere in the text.

- **epsilon** — the Greek letter epsilon, ϵ. Traditionally, ϵ is a very small quantity.

- **Delta** — the Greek letter (capital Delta) Δ. The lower case version is δ. Delta $= \Delta$ usually stands for the change in a quantity. The lower case letter usually refers to a chosen sufficiently small quantity, to a certain type of characteristic function, or to an algebraic function whose formal properties resemble those of the derivative.

- **factorial,** n — $n!$ —The quantity $0!$ is by definition 1. If $n > 0$, then $n! = n(n-1)!$. For example, $3! = 3 \cdot 2 \cdot 1$. The quantity $n!$ counts the number of ways that a set with n distinct elements can be arranged.

- The **greatest integer function**, $\lfloor X \rfloor$, truncates the digits of the real number X that fall to the right of the decimal point.

- **interior of a set** — this is the largest open set contained within a given set.

- The **limit of a real valued sequence of numbers** is denoted by $\lim_{m \to \infty} f(n)$ This limit exists and is equal to an number L if and only if for every $\epsilon > 0$ there is an integer $N > 0$ such that $|f(n) - L| < \epsilon$ whenever $n > N$.

- **logarithm** of a number — the equation $\log_b(y) = x$ is equivalent to the equation $b^x = y$ where y is a positive quantity. The base, b, *must be* a positive real number, and things are uninteresting when $b = 1$. Usually the base is 2 for questions about information, 10 is the common logarithmic base, or e is the natural base.

- A **Matrix** is a rectangular or square array of numbers. An $(m \times n)$-matrix has m rows and n columns. It act by multiplication on an $(n \times 1)$-column vector to give an $(m \times 1)$-column vector. Thus it acts a function from n-space to m-space. Here we use three types of matrices: (1) those that rotate n-space, (2) those that project space to the plane, and (3) identity matrices,I_k, these are $(k \times k)$ square matrices that are substantial blocks of the rotation matrices. The identity matrices have 1s along the diagonal (from upper left to lower right) and 0s elsewhere.
- **mini-simplex** — one of the smaller simplices that is obtained by removing the interior of the convex hull of the midpoints along the edges of a simplex. The term is only used in the context of this book and is not standard.
- **multinomial** — an expression of the form $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ where the exponents $k_1, \ldots, k_n \in \{0, 1, 2, \ldots\}$.
- **multinomial coefficients** — these are the coefficients of the expression $x_1^{k_1} x_2^{k_2} \cdots x_n^{k_n}$ in the expansion of $(x_1 + x_2 + \cdots + x_n)^r$. They are given by the formula
$$\binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n} = \frac{(k_1 + k_2 + \cdots + k_n)!}{k_1! k_2! \cdots k_n!}.$$
In this notation, the binomial coefficient is given as
$$\binom{r}{k} = \binom{r}{k, r-k}.$$
The expression $[k_1, k_2, \ldots, k_n]$ is an internal notation for the multinomial coefficient:
$$[k_1, k_2, \ldots, k_n] = \binom{k_1 + k_2 + \cdots + k_n}{k_1, k_2, \ldots, k_n}.$$
- **the natural numbers** are denoted by $\mathbb{N} = \{1, 2, 3, \ldots\}$. Any two natural numbers can be added, this addition is commutative and associative. It also has a cancelation property: $a + b = a + c$ implies that $b = c$. A key property of the natural numbers is that every number has a successor. This is the basis for mathematical induction. There is also an order relation on the natural numbers.

- **neighborhood,** ϵ — an ϵ neighborhood of a point $\vec{x}$ consists of those points $\vec{y}$ such that the distance between $\vec{x}$ and $\vec{y}$ is strictly smaller than ϵ. **i.e.**

$$N_\epsilon(\vec{x}) = \{\vec{y} : \sqrt{\sum_{j=1}^{n}(x_j - y_j)^2} < \epsilon.$$

  In principal ϵ is chosen to be an arbitrarily small positive real numbers.

- **one-to-one** — a function f is **one-to-one** if $f(x_1) = f(x_2)$ implies $x_1 = x_2$. This property is also called injectivity in the literature.
- **onto** — a function f whose domain is a set X and whose range is Y is **onto** if for every $y \in Y$, there is an $x \in X$ such that f(x)=y. This property is also call surjectivity in the literature
- **open set** — an open set is one in which every point in the set has an ϵ-neighborhood that is contained entirely within the set.
- **Pascal's recursion** is given by the formula:

$$\binom{r}{k} = \binom{r-1}{k} + \binom{r-1}{k-1}.$$

- **rational numbers** — commonly denoted by $\mathbb{Q}$. These are the set of fractions a/b where a and b are integers, and $b \neq 0$. Usually, a and b are chosen so that they have no common factors. More precisely, we say that $a/b = c/d$ if and only if $ad = bc$. Here neither b nor d is 0.
- **real numbers** — denoted $\mathbb{R}$. The real number are the numbers of infinite precision. Any real number can be expressed in decimal notation as a sign ($\pm$), and an infinite string of digits taken from $0, 1, 2, 3, 4, 5, 6, 7, 8, 9$. The string starts on the left, has a decimal to indicate place value and continues infinitely far to the right. We can write, $X = \pm x_n x_{n-1} \ldots x_1 x_0 . x_{-1} x_{-2} \ldots$ where $x_j \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

to mean

$$X = \pm \sum_{k=-n}^{\infty} x_{-k} 10^{-k}.$$

- A **recursion relation** among a sequence of variables is a relationship in which the values of the current element of the sequence are determined by the previous values of the sequence. A recursion relation is not a valid relation unless initial values have been specified.
- The **root of unity** of degree $n+1$ is a complex number ζ_{n+1} that satisfies the equation $(\zeta_{n+1})^{n+1} = 1$ and has the smallest positive angle (as measured from the x-axis) among those that do. This is the point on the unit circle subtended by the angle $\frac{2\pi}{n+1} = 360°/(n+1)$. In complex coordinates, $\zeta_{n+1} = \cos\frac{2\pi}{n+1} + i\sin\frac{2\pi}{n+1}$ where $i^2+1 = 0$ so i is the square root of -1.
- The **sine of an angle** θ, denoted $\sin\theta$ is the y-coordinate of the point on the unit circle that is subtended by the angle θ.
- **theta** the Greek letter theta θ. The capital character is Θ. The lower case letter is the usual name for an unknown angle. Angles are typically measured in radians or degrees. There are by definition 360 degrees in a circle (an arbitrary number but chosen because it is highly divisible and closely approximates the number of days in an earth year). Radian measure is given as the length of an arc on a unit circle that is subtended by an angle θ. A unit circle has radius 1; its circumference is 2π. Thus 2π radians $= 360°$, and angle measurements can be converted from radians to degrees by multiplying by a factor of $180°/\pi$.
- **zeta** the Greek letter zeta ζ. This letter commonly denotes a point on the unit-circle subtended by an angle $\frac{2\pi}{n+1}$.

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