Categorical Quandles and Knots

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May 2010

Knots in Washington XXX
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Plan

1. State Main Results
2. Motivation
3. Categories in Groups and Quandles
4. The Functors Conj and Core
5. Review $SU(2)$
6. Example computations $TSU(2)$
7. Non-group examples
8. The Fundamental 2-quandle.
Main Results

Theorem
The functors Conj and Core, when applied to a category object in the category of groups, give strict 2-quandles.

Theorem
There are strict 2-quandles that don’t come directly from groups.

Theorem
There is a strict 2-quandle that comes from the crossed module structure on $TSU(2)$. $\mapsto$ knot theory invariant.

& some more stuff.
Motivations

Relations via 3-cocycles ?!?!?!
Quandles

Definition
A quandle is a set $X$ that has a binary operation $\triangleleft$ such that
I. $\forall x \in X \quad x \triangleleft x = x$.
II. $\forall x, y \in X \exists! z \in X$ such that $z \triangleleft x = y$. We write $z = y \triangleleft^{-1} x$.
III. $\forall x, y, z \in X \quad (x \triangleleft y) \triangleleft z = (x \triangleleft z) \triangleleft (y \triangleleft z)$. 
Examples of Quandles

1. $G$ is a group, $a, b \in G$, let $a \triangleleft b = b^{-1}ab : \text{Conj.}$

2. $G$ is a group, $a, b \in G$, let $a \triangleleft b = ba^{-1}b : \text{Core.}$

3. $M$ is a $\mathbb{Z}[T, T^{-1}]$-module, let $a \triangleleft b = Ta + (1 - T)b : \text{LX-quandle.}$
Category in $\mathcal{C}$

Here $\mathcal{C}$ is either the category of groups or the category of quandles. If $A$ is an object in $\mathcal{C}$, then its underlying set is denoted by $|A|$. A category in $\mathcal{C}$ is constructed:

- $O, M \in \text{Obj } \mathcal{C}$
- $s, t : M \rightarrow O$ and $i : O \rightarrow M$ are morphisms.
- Note that
  $$M \times_O M = \{(f_2, f_1) : s(f_2) = t(f_1)\} \in \text{Obj } \mathcal{C}.$$  
- $c : M \times_O M \rightarrow M$, composition denoted
  $$c : (f_2, f_1) \mapsto f_2 \circ f_1.$$
Category in $\mathcal{C}$ continued

1. $s(i(x)) = t(i(x)) = x$ for any $x \in O$;
2. $s(f_2 \circ f_1) = s(f_1)$, $t(f_2 \circ f_1) = t(f_2)$;
3. $(f_3 \circ f_2) \circ f_1 = f_3 \circ (f_2 \circ f_1)$;
4. $(i(x) \circ f_1) = f_1$, and $(f_2 \circ i(x)) = f_2$
Suppose that $x, y \in O$. Then
$$\text{hom}(x, y) = \{f \in M : s(f) = x \& t(f) = y\}.$$ 
Given a category $C$, a category in $C$: 
$(O, M, s, t, i, \circ)_C$ or $(O, M, s, t, i, \circ)$
Let $\mathcal{C}_0$ and $\mathcal{C}_1$ denote categories in which the objects have underlying sets. Suppose that $F : \mathcal{C}_0 \to \mathcal{C}_1$ is a functor such that for any object $A$ in $\mathcal{C}_0$, the underlying set $|F(A)|$ is the induced image $F(|A|)$. Denote $F(X) = \hat{X}$.

**Lemma**

If $(O, M, s, t, i, \circ)_{\mathcal{C}_0}$ is a category in $\mathcal{C}_0$, then $(\hat{O}, \hat{M}, \hat{s}, \hat{t}, \hat{i}, \hat{\circ})_{\mathcal{C}_1}$ is a category in $\mathcal{C}_1$. 
Crossed Modules

Definition
A crossed module is a quadruple \((G, H, \alpha, \tau)\), where \(G\) and \(H\) are groups, \(\alpha : G \times H \to H\) defines an action of \(G\) on \(H\), \(\tau : H \to G\) is a group homomorphism.

\[
\alpha(\tau(h), h') = hh'h^{-1},
\]

\[
\tau(\alpha(g, h)) = g\tau(h)g^{-1}.
\]
Examples

1. Let $H \triangleleft G$ denote a normal subgroup. $G$ acts on $H$ by conjugation; $\tau$ is the inclusion.

2. Let $H$ denote a group, and let $G = \text{Aut}(H)$; the map $\tau$ is the inclusion of $H$ in $G$ as inner automorphisms.

3. Another example below.
Crossed Modules $= \text{Cat in group}$

Given $\text{c.mod} (G, H, \alpha, \tau)$, define $(O, M, s, t, i, \circ)_G$

- objects $O = G$,
- morphisms $M = H \rtimes G$ (with multiplication $(h_1, g_1) \cdot (h_2, g_2) = (h_1 \alpha(g_1, h_2), g_1g_2)$),
- source: $s(h, g) = g$,
- target map $t(h, g) = \tau(h)g$,
- the id. $i(g) = (1, g)$,
- comp. $((h_2, \tau(g_1)h_1) \circ (h_1, g_1) = (h_2h_1, g_1)$. 
Apply the Core functor

For morphisms, under conj. we have

\[(h_1, g_1) \triangleleft (h_2, g_2) = (\alpha(g_2^{-1}, h_2^{-1} h_1 \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)\]

\[= (\alpha(g_2^{-1}, h_2^{-1}) \alpha(g_2^{-1}, h_1) \alpha(g_2^{-1}, \alpha(g_1, h_2)), g_2^{-1} g_1 g_2)\]
Consider $g = \begin{pmatrix} x + iy & z + iw \\ -z + iw & x - iy \end{pmatrix}$ where $x^2 + y^2 + z^2 + w^2 = 1$. Let

\[
\hat{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad \hat{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \hat{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix};
\]

then $g = x + y\hat{i} + z\hat{j} + w\hat{k}$. 
More stuff: $su(2)$

Any vector $v \in su(2)$ may be written as $v = a\hat{i} + b\hat{j} + c\hat{k}$. Observe $\hat{i} \cdot \hat{j} = \hat{k}$, etc.

In particular, $\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = -1$.

Standard basis: $\mathbf{i} = \hat{i}/2$, $\mathbf{j} = \hat{j}/2$, and $\mathbf{k} = \hat{k}/2$

Brackets: $\mathbf{i} \times \mathbf{j} = [\mathbf{i}, \mathbf{j}] = \mathbf{k}$, and $\mathbf{i} \times \mathbf{i} = 0$. 
Conjugate an algebra element by 
\[ g = x + y\hat{i} + z\hat{j} + w\hat{k} \in SU(2). \] The matrix of this rotation (in the ordered basis \((\hat{i}, \hat{j}, \hat{k})\)) is

\[
\begin{pmatrix}
  x^2 + y^2 - z^2 - w^2 & 2(yz - xy) & 2(wy + xz) \\
 2(yz + xw) & x^2 + z^2 - y^2 - w^2 & 2(wz - xy) \\
2(yw - xz) & 2(xy + zw) & x^2 + w^2 - y^2 - z^2
\end{pmatrix}.
\]

The axis is \( y\hat{i} + z\hat{j} + w\hat{k} \). Let \( x = \cos(\theta) \), then the angle of rotation about this axis is \( \pm 2\theta \).
The tangent bundle of $SU(2)$

In general, a Lie group and its Lie alg. define a crossed module. Here $G = SU(2)$, acts upon the abelian group $H = su(2)$ (vector space) via conj. $\alpha(g, h) = ghg^{-1}$. And $\tau : H \to G$ is (guesses?) the trivial map.
So in the corresponding category, $M = TSU(2)$. 
The binary dihedral group

Consider the Hopf link $\in SU(2)$.

$$S_{1,i}^1 = \{ e^{i\theta} \} \cup S_{j,k}^1 = \{ e^{i\phi} \hat{j} \}.$$ 

This link is a subgroup of $SU(2)$. A representation of the fundamental group of a 2-bridge knot into $SU(2)$ is conjugate to one that lands upon $S_{j,k}^1$. We may pick one generator to land at $\hat{j}$. 
Consequently, we need to know ... 

\[(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}, \cos(s) \hat{j} + \sin(s) \hat{k})\]
\[\triangleleft (a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}, \cos(t) \hat{j} + \sin(t) \hat{k})\]
\[= (2a_2 - a_1) \hat{i}\]
\[+ b_1 \cos(2t)\]
\[+ 2 \sin(s - 2t)(c_2 \cos(s) - b_2 \sin(s)) + c_1 \sin(2t)] \hat{j}\]
\[+ [\sin(2t)(b_1 + b_2 \cos(2s) - b_2 + c_2 \sin(2s))\]
\[+ \cos(2t)(-b_2 \sin(2s) - c_1 + c_2 \cos(2s) + c_2)] \hat{k},\]
\[\cos(2t - s) \hat{j} + \sin(2t - s) \hat{k})\]
Example, the right-handed trefoil

Up to conjugation there is one representation into $SU(2)$ in which one arc is colored $\hat{j}$. In contrast, when coloring by $su(2) \rtimes SU(2)$,

- Left top arc $(a_1 \hat{i} + b_1 \hat{j} + c_1 \hat{k}, \hat{j})$,
- Right top arc $(a_2 \hat{i} + b_2 \hat{j} + c_2 \hat{k}, \cos(t)\hat{j} + \sin(t)\hat{k})$.
- $t = \pm 2\pi/3$
- $a_1 = a_2$,
- $b_1 = \frac{1}{2} (-b_2 + \sqrt{3}c_2)$.
- center arc next page
\[
\begin{align*}
(a_2 \hat{i} + \left( \frac{-b_2 + \sqrt{3}c_2}{4} + \frac{\sqrt{3}}{2}(2c_2 - c_1) \right) \hat{j} \\
- \left( \frac{1}{2}(2c_2 - c_1) + \frac{\sqrt{3}}{4}(-b_2 + \sqrt{3}c_2) \right) \hat{k},
\end{align*}
\]
\[
\left( -\frac{1}{2} \hat{j} - \frac{\sqrt{3}}{2} \hat{k} \right).
\]
So the space of quandle representations for the trefoil into $TSU(2)$ has 4 free parameters. We have a similar computation for the figure eight knot.
Let $A$ denote a $\mathbb{Z}[T, T^{-1}]$-module. Let $O = A$; let $M = A \times A$.

- On $O$ we have $a \triangleleft b = Ta + (1 - T)b$;
- On $M$, we have $(a_1, a_2) \triangleleft (b_1, b_2) = (Ta_1 + (1 - T)b_1, Ta_2 + (1 - T)b_2)$.
- $i(a) = (0, a)$;
- $s(a, b) = b$;
- $t(a, b) = a + b$.
- $(a, c + d) \circ (c, d) = (a + c, d)$.

This will be called an *Alexander 2-quandle*.
Fundamental strict 2-quandle

Let $K$ denote an $n$-mfd. embedded in an $(n + 2)$-mfd. $M$. We define a 2-quandle

$$\pi^{(2)}_Q(K) = \pi^{(2)}_Q(K, M) :$$

- The quandle of objects: $O = \pi_Q(K)$, (describe)
- The quandle of morphisms: $M$ hmtpy classes $a \cup \gamma \cup b$, where $a, b$ are arcs “$\in \pi_Q(K)$.” $\gamma$ is an oriented arc between the feet of $a$ and $b$. 

![Diagram of a 2-quandle](image-url)
Fundamental strict 2-quandle

- \((b, \gamma, a) \in M\) — note: really an equiv. class.
- source: \(s(b, \gamma, a) = a\).
- target: \(t(b, \gamma, a) = b\).
- id: \(i(a) = (a, c, a)\) where \(c\) is the constant arc.
- Composition:
  \[
  (a_2, \gamma_2, a_1) \circ (a_1, \gamma_1, a_0) = (a_2, \gamma_2 \circ \gamma_1, a_0)
  \]
Fundamental strict 2-quandle

Quandle operations

- On objects: same as $\pi_Q(K)$
- On morphisms:
  $$(a_1, \gamma, a_0) \diamond (b_1, \delta, b_0) = (a_1 \diamond b_1, \gamma, a_0 \diamond b_0)$$
Closing remarks

- We have a working def of a strict 2-quandle.
- We don’t have the non-strict case, but we hope this is coordinated by quandle 3-cocycles.
- We have some interesting parameters when we extend representations to $TSU(2)$.
- The fund. 2-quandle does not appear to give more info. in the classical case.
- Masahico will discuss the virtual case as well as presentations for the fund. 2-quandle.