Cohomology in self-distributive coalgebras and elsewhere

Carter
Crans
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Outline

1. Statements of main results
2. Quandle Cohomology
3. Diagrammatics of Algebraic Relations
4. Hopf Algebras
5. Deformation cocycles in Hopf alg.
6. Analogous construction in cocommutative coalgebras
Theorem

There is a cohomology theory in which both quandle cocycles and Lie algebra cocycles give non-trivial cocycles.
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<thead>
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- [SD] $(a \triangleright b) \triangleright c = (a \triangleright c) \triangleright (b \triangleright c)$

Think: Group with $a \triangleright b = bab^{-1}$. 
Statements of main results

Quandle Cohomology

Diagrammatics of Algebraic Relations

Hopf Algebras

Deformation cocycles in Hopf alg.

Analogous construction in cocommutative coalgebras

Carter Crans Elhamdadi Saito

Cohomology in self-distributive coalgebras and elsewhere
Cocycles

**Definition**

A map \( \phi : X \times X \to A \) into an (additive) abelian group \( A \) is a *quandle 2-cocycle* if

\[
\phi(a, b) + \phi(a \triangledown b, c) = \phi(a, c) + \phi(a \triangledown c, b \triangledown c).
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Cocycles

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A map $\phi : X \times X \to A$ into an (additive) abelian group $A$ is a **quandle 2-cocycle** if

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\phi(a, b) + \phi(a \triangleleft b, c) = \phi(a, c) + \phi(a \triangleleft c, b \triangleleft c).
$$

A 3-cocycle $\theta$ satisfies

$$
\theta(a, b, c) + \theta(a \triangleleft b, c, d) + \theta(a, b, d) = \theta(a \triangleleft d, b \triangleleft d, c \triangleleft d) + \theta(a, c, d) + \theta(a \triangleleft c, b \triangleleft c, d).
$$
Applications of Quandle Cocycles

- Obstructions to Tangle Embeddings [Ameur, Saito]
Applications of Quandle Cocycles

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- [Kamada, Iwakiri, Jelsovsky, Nelson, Ishii, JSC]
- Eisermann identified the quandle 2-cocycle inv. with a coloring polynomial inv. $\sum_{\text{reps}} \text{im} \lambda$. 

Carter Crans Elhamdadi Saito Cohomology in self-distributive coalgebras and elsewhere
Relations to Lie Algebras

Let $\mathfrak{g}$ denote a Lie Alg. over a field $\mathbb{F}$. 

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Then $\mathbb{F} \oplus \mathfrak{g}$ is a cocommutative coalgebra with counit $\Delta(x) = x \otimes 1 + 1 \otimes x$ for $x \in \mathfrak{g}$, $\Delta(1) = 1 \otimes 1$ and $\epsilon(x) = 0$ for $x \in \mathfrak{g}$. 

The function $q : \mathbb{F} \otimes \mathfrak{g} \otimes \mathbb{F} \to \mathfrak{g}$ defined by $q((a, x) \otimes (b, y)) = (ab, bx + [x, y])$ satisfies a coalgebra self-distributive relation and can be used to define a solution to the Yang-Baxter relation. [Woronowicz] Studied by Crans in her dissertation.
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\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = \]
Lie alg, self distrib, and YBE

\[ q(q \otimes 1)((a, x) \otimes (b, y) \otimes (c, z)) = q((ab + bx + [x, y]) \otimes (c, z)) = abc + bcx + c[x, y] + b[x, z] + [[x, y], z] \]
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\[ q((ac + cx + [x, z]) \otimes (b + y)) + q((a + x) \otimes [y, z]) = \]
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(abc + bcx + c[x, y] + b[x, z] + [[x, z], y]) + [x, [y, z]] \]

\[ R_q((a, x) \otimes (b, y)) = (b, y) \otimes (a, x) + (1, 0) \otimes (0, [x, y]) \]
Remarks

Crans’s dissertation discusses this in relation to her construction of so-called Lie 2-algebras. In particular, the Lie cocycle, $\langle a, [b, c] \rangle$ is used to construct a solution to the Zamolodchikov equation.
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Algebra

Assoc. \[=\] \[=\]

Unit \[=\] \[=\]
Coalgebra

\[ Y = \triangleleft \triangleleft \]

Co-associativity

\[ Y = \big| = \big| \]

Co-unitality
Hopf Algebra

- Associativity
- Coassociativity
- Compatibility
- Unit
- Counit
- Unit is a coalgebra hom
- Counit is an algebra hom
- Antipode condition

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Cohomology in self-distributive coalgebras and elsewhere
For example the adjoint map is given by:
Example: group algebra

- $G$ is a group
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a, b) = ab$
Example: group algebra

- $G$ is a group and $\mathbb{F}[G]$ is its group alg. over a field $\mathbb{F}$
- $\mu(a, b) = ab$
- $\Delta(a) = a \otimes a$
Example: group algebra

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\[ G \text{ is a group and } \mathbb{F}[G] \text{ is its group alg. over a field } \mathbb{F} \]
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In this case, the adjoint map is given on basis elements by $ad(a \otimes b) = b^{-1}ab$. 
Two important properties

(A)  

(B)
Two important properties

(A) and (B)
Woronowicz sol’n to YBE
\[ d^{1,1}(\text{\textbullet}) = \begin{array}{c}
\text{\textbullet} \\
\end{array} - \begin{array}{c}
\text{\textbullet}
\end{array} + \begin{array}{c}
\text{\textbullet}
\end{array} \]
\[ d^{1,1}(\emptyset) = \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \\ \text{Diagram 3} \end{array} \]

\[ \eta_1 = \begin{array}{c} \text{Diagram 4} \end{array} , \quad d^{2,1}(\begin{array}{c} \text{Diagram 5} \end{array}) = \begin{array}{c} \text{Diagram 6} + \text{Diagram 7} - \text{Diagram 8} = 0 \end{array} , \quad d^{2,2}(\begin{array}{c} \text{Diagram 9} \end{array}) = \begin{array}{c} \text{Diagram 10} - \text{Diagram 11} = 0 \end{array} \]
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\[ d^{1,1}(\begin{array}{c}
\end{array}) = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]

\[ \eta_1 = \begin{array}{c}
\end{array}, \quad d^{2,1}(\begin{array}{c}
\end{array}) = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = 0, \quad d^{2,2}(\begin{array}{c}
\end{array}) = \begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} = 0 \]

\[ d^{2,1}(\begin{array}{c}
\end{array}) = \left(\begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array}\right) + \left(\begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array}\right) - \left(\begin{array}{c}
\end{array} - \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array}\right) = 0 \quad \text{if} \quad \begin{array}{c}
\end{array} = \begin{array}{c}
\end{array} + \begin{array}{c}
\end{array} \]
\[ d^{1,1} ( \circ ) = \Delta \circ - \Delta + \emptyset \]

\[ \eta_1 = \uparrow, \quad d^{2,1} ( \uparrow ) = \Delta \uparrow + \Delta \uparrow - \lambda = 0, \quad d^{2,2} ( \uparrow ) = \lambda \Delta - \lambda \Delta = 0 \]

\[ d^{2,1} ( \uparrow ) = \left( \Delta - \lambda + \lambda \right) + \left( \lambda - \lambda + \Delta \right) - \left( \Delta - \lambda + \lambda \right) = 0 \quad \text{if} \quad \lambda = \lambda + \lambda \]

\[ d^{2,2} ( \uparrow ) = \left( \left( \Delta - \lambda + \lambda \right) + \lambda \lambda \right) - \left( \left( \lambda - \lambda + \Delta \right) - \lambda \lambda \right) + \lambda \lambda \]

\[ = \left( \left( \left( \Delta - \lambda + \lambda \right) - \lambda \lambda \right) + \lambda \lambda \right) - \left( \left( \lambda - \lambda + \Delta \right) - \lambda \lambda \right) + \lambda \lambda \]

\[ = \left( \left( \Delta - \lambda \lambda + \lambda \lambda \right) - \lambda \lambda \right) - \left( \lambda - \lambda + \Delta + \lambda \lambda \right) = 0 \quad \text{if} \quad \lambda + \lambda = \lambda + \lambda \]
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\( \xi_1 \)

\( \xi_2 \)

(A)

(B)

(C)
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1

(C)

(A)

(B)

(C)

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Cohomology in self-distributive coalgebras and elsewhere
\[ d^{3,1} \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} + \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} - \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} - \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \]

\[ = \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} + \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} - \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} - \begin{pmatrix} \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet \end{pmatrix} \]
$d^{3,1} = 
\begin{align*}
\ &+ \quad \quad \quad + \quad \quad \quad + \\
\ &= \left( \begin{array}{c}
\text{triangle}
\end{array} \right) + \left( \begin{array}{c}
\text{triangle}
\end{array} \right) + \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right)
\end{align*}$

\begin{align*}
\ &= \left( \begin{array}{c}
\text{triangle}
\end{array} \right) + \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right) - \left( \begin{array}{c}
\text{triangle}
\end{array} \right)
\end{align*}
\[ d^{3,1} \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} = \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} + \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \]

\[ = \left( \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} + \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \right) + \left( \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} + \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \right) - \left( \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} + \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \right) - \left( \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} + \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} - \begin{pmatrix} \uparrow \\ \downarrow \end{pmatrix} \right) \]
Statements of main results
Quandle Cohomology
Diagrammatics of Algebraic Relations
Hopf Algebras
Deformation cocycles in Hopf alg.
Analogous construction in cocommutative coalgebras

\[ d^{3,1} \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) = \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) + \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) \]

\[ = \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) + \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) + \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) \]

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\[ + \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) = \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) + \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) - \left( \begin{array}{c} \bullet \\ \longrightarrow \\ \longrightarrow \end{array} \right) \]

Carter Crans Elhamdadi Saito
Cohomology in self-distributive coalgebras and elsewhere
Self-distributive coalg

Example

Let $X$ be a finite set with a self-distributive operation.
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Let $X$ be a finite set with a self-distributive operation. Consider $F[X]$. 

Set $W = F \oplus F[X]$. 

$\Delta(x) = x \otimes x$, for basis elements $x \in X$ while $\Delta(1) = 1 \otimes 1$. 

$\epsilon(x) = 1$ while $\epsilon(1) = 1$. 

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- $\epsilon(x) = 1$ while $\epsilon(1) = 1$. 
Define $q : W \otimes W \to W$ by linearly extending $q(x \otimes y) = x \triangleleft y$, $q(1 \otimes x) = 1$, $q(x \otimes 1) = 0$, and $q(1 \otimes 1) = 0$. More explicitly,

$$q( (a + \sum a_x x) \otimes (b + \sum b_y y) ) = \sum y a b_y + \sum x, y a_x b_y (x \triangleleft y).$$

This is a self-distributive operation that is compatible with comultiplication.
Theorem

For a quandle 2-cocycle $\phi$ with the coefficient group $A = \mathbb{F}$, define $\hat{\phi} : W \otimes W \to W$ by linearly extending $\hat{\phi}(x \otimes y) = \phi(x, y)$, $\hat{\phi}(1 \otimes x) = 1$, and $\hat{\phi}(x \otimes 1) = \hat{\phi}(1 \otimes 1) = 0$ for $x, y \in X$. Then $\hat{\phi}$ is a cocycle in the analogous deformation theory of self-distributive cocommutative coalgebras.

Theorem

If $\phi$ is not a coboundary, then $\hat{\phi}$ is not a coboundary. In particular, if $H^2_Q(X; k) \neq 0$, then $H^{2,1}_{sh}(W; W) \neq 0$. 
Theorem

Let $\psi : g \times g \to g$ be a Lie algebra 2-cocycle with adjoint action. Define $\hat{\psi} : N \otimes N \to N$ by $\hat{\psi}((a + x) \otimes (b + y)) = \psi(x \otimes y)$ for $a, b, c \in k, x, y, z \in g$. Then $\hat{\psi}$ is a shelf 2-cocycle.