Cohomology in self-distributive coalgebras and elsewhere

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Crans
Elhamdadi
Saito

August 2007

Outline

State results, lest I forget

Theorem

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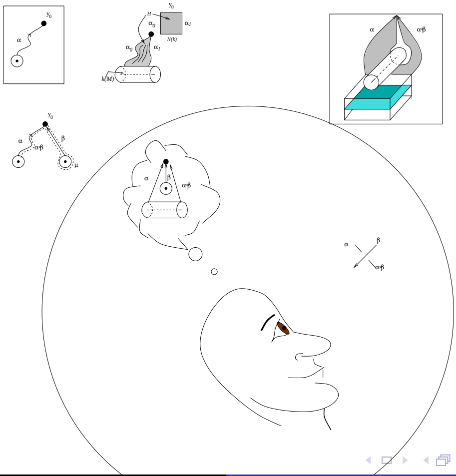
Think: Group with $a \triangleleft b = bab^{-1}$.


y_0
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a 3-cocycle θ satisfies

$$\theta(a, b, c) + \theta(a \triangleleft b, c, d) + \theta(a, b, d)$$

$$= \theta(a \triangleleft d, b \triangleleft d, c \triangleleft d) + \theta(a, c, d) + \theta(a \triangleleft c, b \triangleleft c, d).$$

Applications of Quandle Cocycles

- Obstructions to Tangle Embeddings [Ameur, Saito]

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- [Kamada, Iwakiri, Jelsovsky, Nelson, Ishii, JSC]
- Eisermann identified the quandle 2-cocycle inv. with a coloring polynomial inv. $\sum_{\text{reps}} \text{im}\lambda$.

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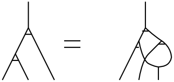

Lie alg, self distrib, and YBE

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$$R_q((a,x) \otimes (b,y)) = (b,y) \otimes (a,x) + (1,0) \otimes (0,[x,y])$$

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Crans's dissertation discusses this in relation to her construction of so-called Lie 2-algebras. In particular, the Lie cocycle, $\langle a, [b, c] \rangle$ is used to construct a solution to the Zamolodchikov equation.

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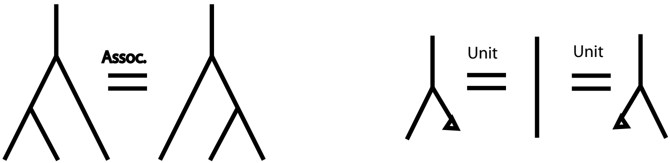
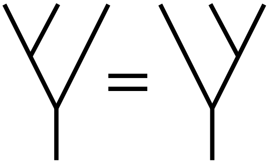
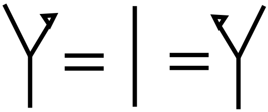
Algebra



Coalgebra

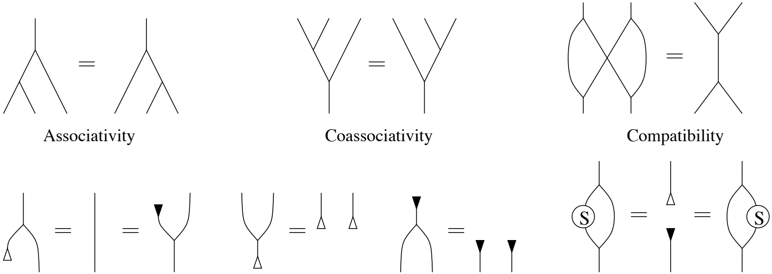


Co-assoc.



Co-unit

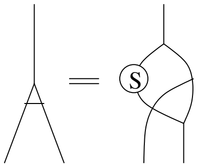
Hopf Algebra



Associativity — Coassociativity — Compatibility

Unit — Counit — Unit is a coalgebra hom — Counit is an algebra hom — Antipode condition

For example the adjoint map is given by:



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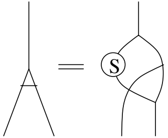
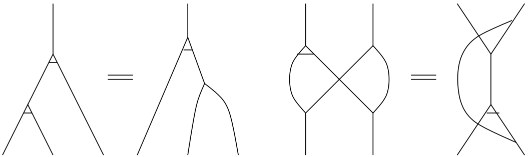
In this case, the adjoint map is given on basis elements by $ad(a \otimes b) = b^{-1}ab$.

Two important properties



(A)



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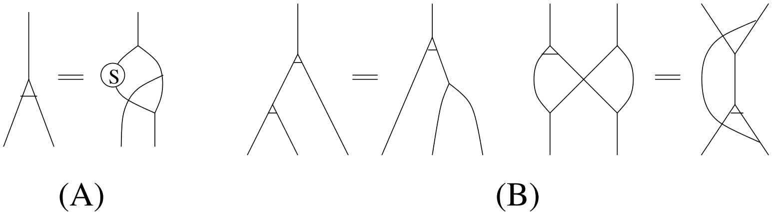
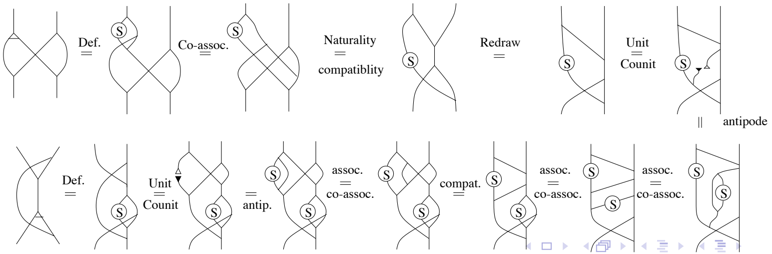
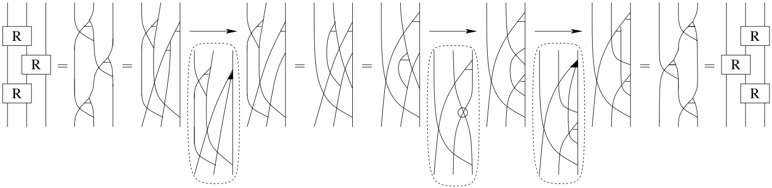
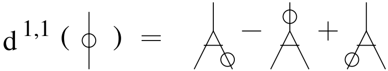




Woronowicz sol'n to YBE

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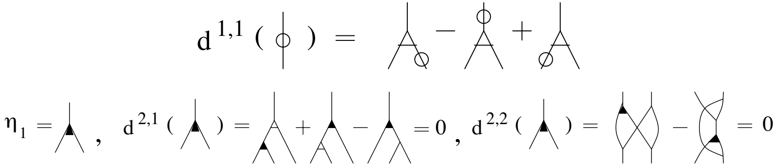
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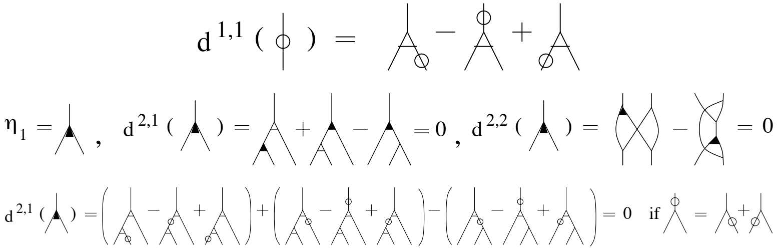
= 0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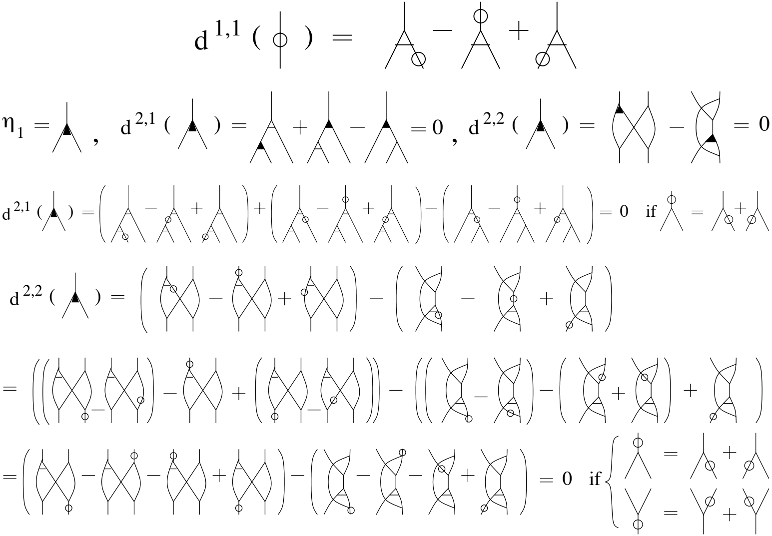
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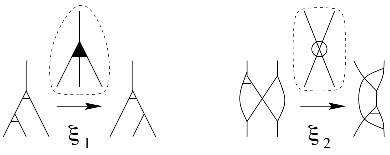
ξ 1
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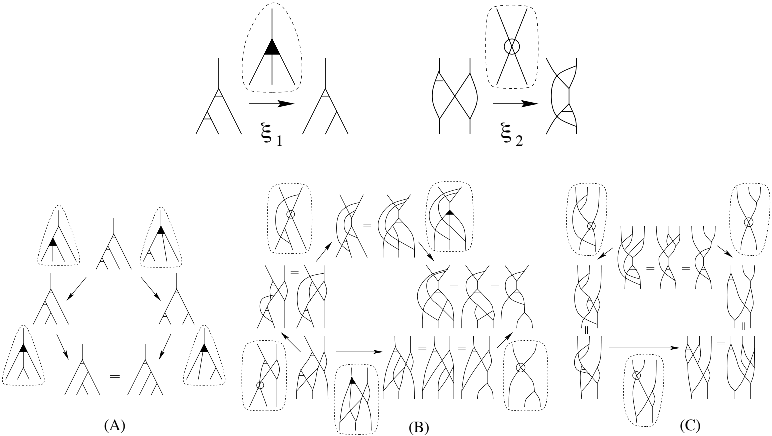
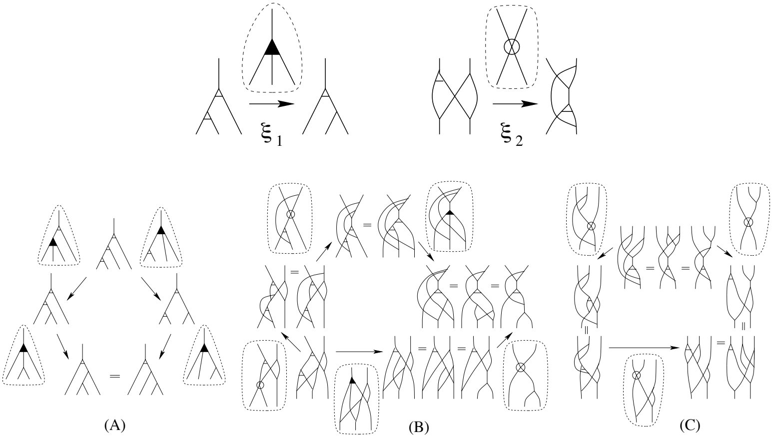

ξ 1
ξ 2
(A)
(B)
(C)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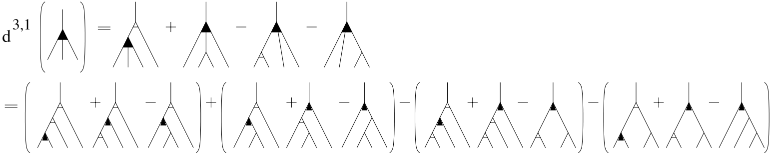
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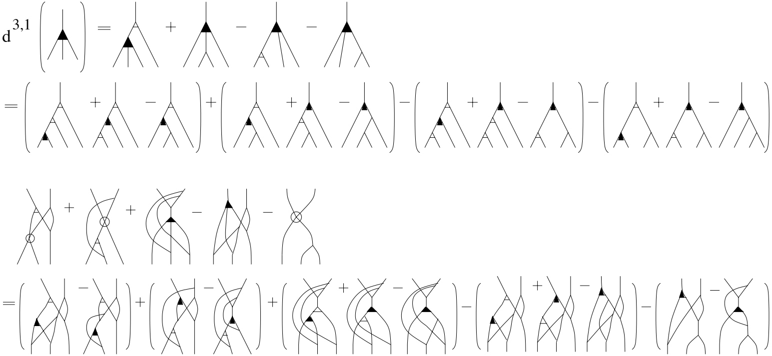
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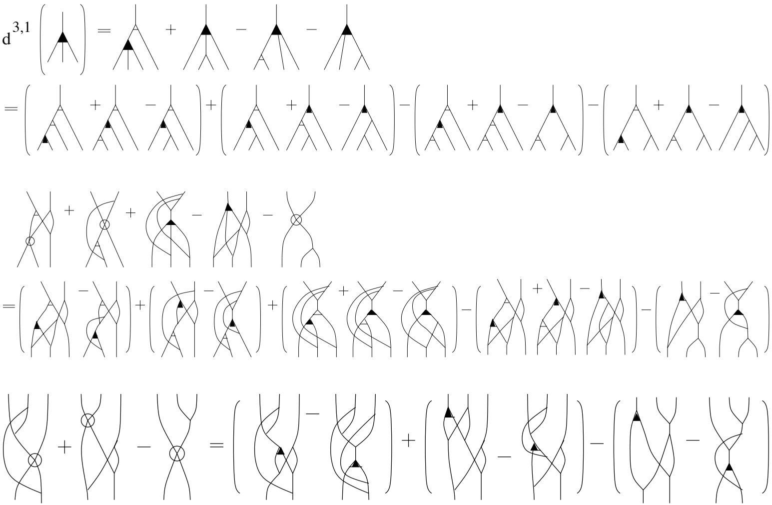
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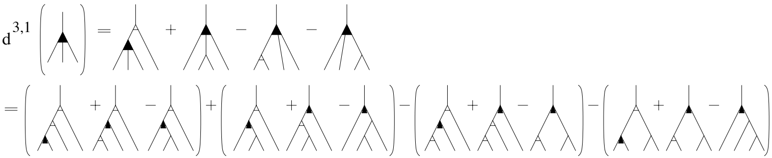
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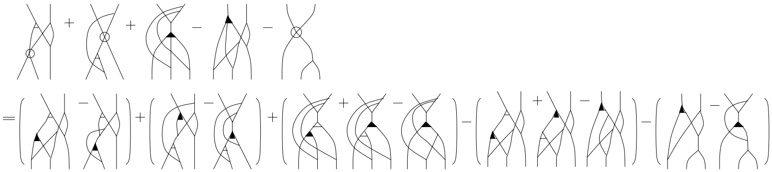

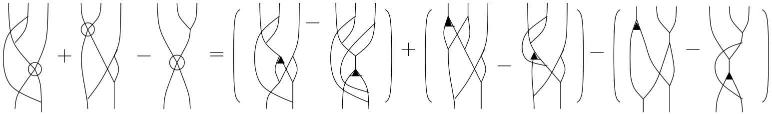

Self-distributive coalg

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- $\epsilon(x) = 1$ while $\epsilon(1) = 1$.

Define $q : W \otimes W \rightarrow W$ by linearly extending $q(x \otimes y) = x \triangleleft y$, $q(1 \otimes x) = 1$, $q(x \otimes 1) = 0$, and $q(1 \otimes 1) = 0$. More explicitly,

$$q(\ (a + \sum a_x x) \otimes (b + \sum b_y y) \ ) = \sum_y ab_y + \sum_{x,y} a_x b_y (x \triangleleft y).$$

This is a self-distributive operation that is compatible with comultiplication.

Theorem

For a quandle 2-cocycle ϕ with the coefficient group $A = \mathbb{F}$, define $\hat{\phi} : W \otimes W \to W$ by linearly extending $\hat{\phi}(x \otimes y) = \phi(x, y)$, $\hat{\phi}(1 \otimes x) = 1$, and $\hat{\phi}(x \otimes 1) = \hat{\phi}(1 \otimes 1) = 0$ for $x, y \in X$. Then $\hat{\phi}$ is a cocycle in the analogous deformation theory of self-distributive cocommutative coalgebras.

Theorem

If ϕ is not a coboundary, then $\hat{\phi}$ is not a coboundary. In particular, if $H^2_{\mathrm{Q}}(X; k) \neq 0$, then $H^{2,1}_{\mathrm{sh}}(W; W) \neq 0$.

Theorem

Let $\psi : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra 2-cocycle with adjoint action. Define $\hat{\psi} : N \otimes N \to N$ by $\hat{\psi}((a + x) \otimes (b + y)) = \psi(x \otimes y)$ for $a, b, c \in k$, $x, y, z \in \mathfrak{g}$. Then $\hat{\psi}$ is a shelf 2-cocycle.