

Non-orientable surface knots that have an arbitrarily  
large number of triple points in their projections

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# Plan

1. Background
2. Main Result
3. Quandles
4. Symmetric Quandle Homology
5. The Quandle  $QS_6$
6. Cool Computations
7. Diagram of the Example

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The *triple point number* of a surface link is the smallest number of triple points among all diagrams of the link.

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The triple points are analogous to crossing points. So the triple pt.  $\#$  is analogous to the crossing number.

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The example given here is the first connected non-orientable surface with triple point bounds.

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*For any positive integer  $N$ , there is a closed 3-manifold  $M$  and a non-orientable surface-knot  $F$  in  $M \times [0, 1]$  such that  $t(F) > N$ .*

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- $(X, \rho)$  sym. quand., an  $(X, \rho)$ -set is a set  $Y$  w/ a rt. action of  $G_{(X,\rho)}$ .

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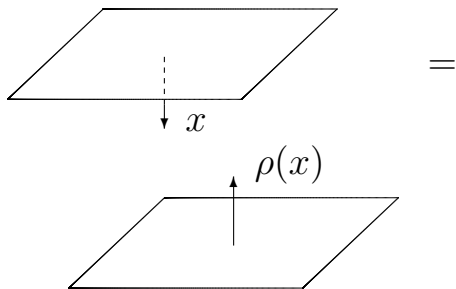
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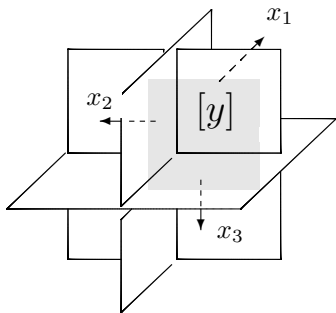
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$$C_n^{Q,\rho}(X)_Y = C_n(X)_Y / (D_n^Q(X)_Y + D_n^\rho(X)_Y).$$

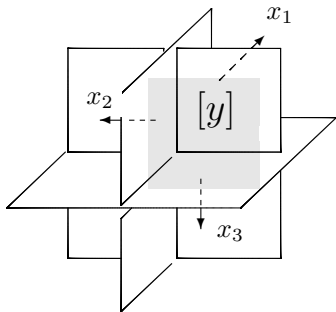


The local orientation can be reversed at the expense of applying the good involution.



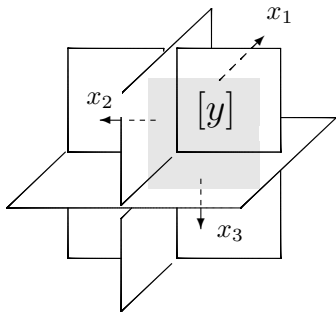
$$+(y, x_1, x_2, x_3)$$

Choose a region.



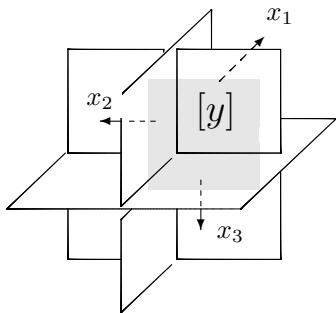
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Choose a region. Get T,M,B normals outward.



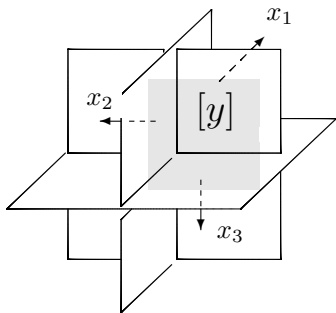
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 Make the chain  $(b, m, t)$  be the colors here.



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 Adjust for sign. Eval. sym. 3-cocycle on this  
 chain.

## Lemma

### **Kamada, Kamada-Oshiro**

*Let  $\theta : \mathbb{Z}(X^3) \rightarrow \mathbb{Z}$  be a sym. quan. 3-coc. s.t.  
 $\theta(a, b, c) \in \{0, -1, 1\} \forall (a, b, c) \in X^3$ . If  
 $\theta([C_D]) = \alpha \in \mathbb{Z}$ , then  $t(F) \geq |\alpha|$ .*

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The quandle that we need:  $\tilde{R}_3 = QS_6$

$R \triangleleft C$	0	1	2	3	4	5
0	0	5	1	0	2	4
1	2	1	3	5	1	0
2	4	0	2	1	3	2
3	3	2	4	3	5	1
4	5	4	0	2	4	3
5	1	3	5	4	0	5

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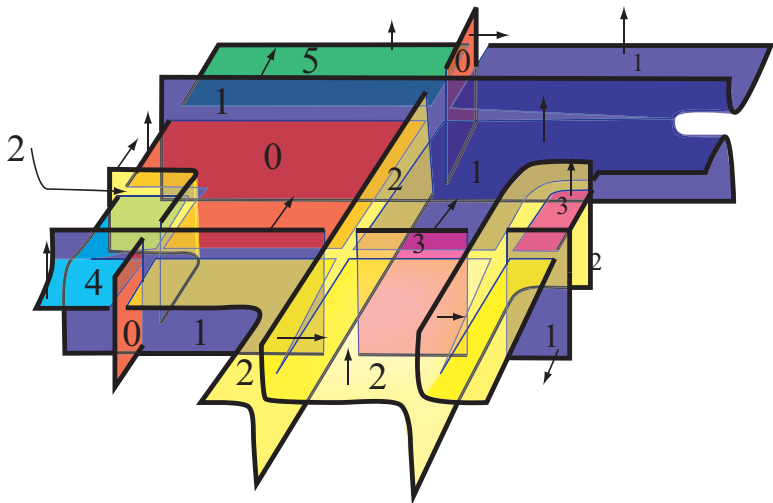
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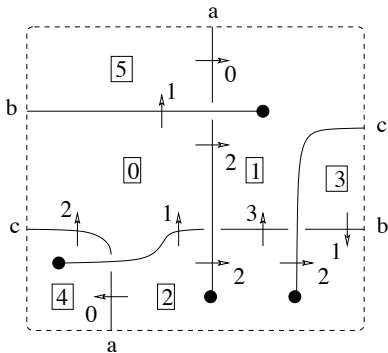
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- (ii) The 3-chain  $c = (2, 1, 2) + (2, 0, 1) - (1, 0, 2) - (0, 2, 1) \in C_3^{Q,\rho}(\tilde{R}_{2n+1}, \mathbb{Z})$  represents a generator of  $H_3^{Q,\rho}(\tilde{R}_3, \mathbb{Z})$ .
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# Ta Daaaaa



# Easier to parse



Thank you California.