Math 125 Carter Final Exam Fall 2010

General Instructions. Work all of your problems and demonstrate your work in your blue books. Do not write your name inside your blue book. Please insert this test into the blue book as you leave. Write neat complete solutions to the problems below.

Cole slaw is best made with a hand shredder. A pair of medium size carrots and a half a head of cabbage are necessary. Optional ingredients include an onion. Shred carrots, (onion) and carrots in a large mixing bowl. Add an ample amount of mayonnaise, a couple of tablespoons of sugar, and the same amount of vinegar. Add sweet pickle relish or chow-chow. Mix thoroughly with a soft spatula.

1. (5 points) Give the $\epsilon, \delta$ definition of the phrase:

$$\lim_{x \to c} f(x) = L.$$ 

2. (5 points) Use the definition of the derivative: (a function $y = f(x)$ is differentiable at $x = c$ provided the limit

$$\lim_{h \to 0} \frac{f(c + h) - f(c)}{h}$$
exists) to prove the product rule:

$$(fg)'(x) = f(x)g'(x) + f'(x)g(x).$$

3. (5 points) Use the fact $\lim_{x \to 0} \frac{\sin(x)}{x} = 1$ to prove that

$$\lim_{x \to 0} \frac{\cos(x) - 1}{x} = 0.$$ 

4. (5 points) Prove that $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

5. (5 points each) Compute the limits:

(a) $$\lim_{h \to 0} \frac{1}{h} \left( \frac{1}{x + h} - \frac{1}{x} \right)$$
6. Compute $f'(x)$ or $\frac{dy}{dx}$ of the following (5 points each):

(a) $f(x) = 3x^4 - 2x^3 + 3x - 2$

(b) $y = \cos^2(x)$

(c) $f(x) = e^{(x\sin(x))}$

(d) $f(x) = \frac{1+2x-x^2}{(x^2+x)^2}$

(e) $f(x) = \int_{x}^{x^2} t \cos(t^2)dt$

(f) $y + \frac{x}{y} = 1$

7. (5 points) Use differentials to estimate $\sqrt{24.8}$.

8. (10 points) Coffee drips at a rate of 5 cubic centimeters per second from a 18 centimeter tall Melita cone whose radius is 9 centimeters. How fast is the height of the coffee decreasing when the height is 5 centimeters? The volume of a cone is $V = \frac{\pi}{3}r^2h$.

9. (10 points) A rectangle is inscribed with two vertices on the parabola $y = 48 - x^2$ and with its base along the $x$-axis. What are the dimensions of the rectangle of maximal area that fits this description?
10. *(10 points each)* Sketch the graphs of the following functions.

(a) 
\[ f(x) = 3x^2 - 4x - 8 \]

(b) 
\[ f(x) = \frac{x - 1}{x^2 + x} \]

(c) 
\[ f(x) = 2 \sin (3x - \pi) \]

for \( x \in [0, 2\pi] \).

11. *(5 points each)* Compute the following anti-derivatives and definite integrals

(a) 
\[ \int_{1}^{4} x^3 dx \]

(b) 
\[ \int \frac{(x + 1)}{\sqrt{x}} dx \]

(c) 
\[ \int \frac{dx}{2x} \]

(d) 
\[ \int e^{2x} dx \]

(e) 
\[ \int_{-\pi/2}^{\pi/2} \cos(x) \, dx \]