Math 227 Carter Sample for test 3

By popular request, I am preparing a sample study guide. The first question is familiar, but now more questions are attached. A problem such as this will be a substantial portion of the test. The remaining integrals should be as straight forward, but not identical to those listed here.

1. Consider the quadratic surface \( f(x, y) = (x - 5)^2 - (y - 12)^2 \).
   
   (a) Sketch the \( z = 0 \), \( z = -1 \), and \( z = 1 \) levels of the surface.
   
   (b) Compute the gradient \( \vec{\nabla} f \).
   
   (c) Find the critical point(s) of \( f(x, y) \).

   (d) Compute the determinant of the Hessian \( H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \) at the critical point.

   (e) Is the critical point a local maximum, minimum, or neither? Explain why?

   (f) Sketch the gradient vector field at appropriate points.

   (g) Compute the work done in moving a particle once around a circle \( x^2 + y^2 = 4 \) in the gradient field.

2. Calculate the volume that is enclosed by the sphere \( x^2 + y^2 + z^2 = 25 \) and the cylinder \( x^2 + y^2 \leq 1 \).

3. Set up an integral that computes the surface area of the region of a sphere \( x^2 + y^2 + z^2 = a^2 \) that lies above the circle \( r = a \cos(\theta) \).

4. Compute the line integral \( \int_C zdx + xdy + ydz \) over the curve \( x = t^2, y = t^3, \) and \( z = t^2 \) for \( t \in [0, 1] \).

5. Determine whether or not \( \vec{F} \) is conservative. If so, find a potential function.

   \[ \vec{F}(x, y) = (6x + 5y)\mathbf{i} + (5x + 6y)\mathbf{j} \]