Math 227 Carter Test 1 Fall 2006

Do all your work in your blue books. Write your solutions in your blue book. Show all work. Write your name on only the outside of your blue book. Write neatly, and use complete sentences when appropriate. My hope is that you do well on this exam. You may receive more than 100 points of credit. Good luck.

1. Consider the quadratic surface \( f(x, y) = (x - 3)^2 + (y - 4)^2 \) (5 points each item).
   (a) Sketch the \( z = 25 \) level of the surface.
   (b) Compute the gradient \( \vec{\nabla} f \).
   (c) Find the critical point(s) of \( f(x, y) \).
   (d) Compute the determinant of the Hessian \( H = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{pmatrix} \) at the critical point.
   (e) Is the critical point a local maximum, minimum, or neither? Explain why?
   (f) Compute the directional derivative of \( f(x, y) \) at \((0, 0)\) in the direction \((\sqrt{2}/2, \sqrt{2}/2)\).
   (g) At the point \((6, 8)\) in which direction does the gradient point?
   (h) Compute the equation of the plane tangent to the surface at the point \((0, 0)\).

2. (10 points) Find and classify the critical point(s) of the function \( f(x, y) = x^3 y + 12x^2 - 8y \).

3. (10 points) Identify the surface in 3-space that is represented by the equation \( r = 4 \sin(\theta) \) in spherical/polar coordinates.

4. (10 points) Find the point on the ellipsoid \( x^2 + 2y^2 + 3z^2 = 1 \) at which the tangent is parallel to the plane \( 3x - y + 3z = 1 \).

5. (10 points) Let \( f(x) = e^x \cos(x) \) and \( g(x) = \sqrt{x^2 + 1} \). Show that \( z = F(x, t) = f(x + 2t) + g(x - 2t) \) satisfies the wave equation
   \[
   \frac{\partial^2 z}{\partial x^2} = 4 \frac{\partial^2 f}{\partial t^2}.
   \]

6. (10 points) Calculate
   \[
   \int_0^2 \int_0^1 (2x + y)^8 \, dx \, dy.
   \]

7. (10 points) Set up the double integral that will compute the volume of the tetrahedron that is bounded in the totally positive octant by the coordinate planes \( x = 0, \ y = 0, \ z = 0, \) and the plane \( x + y + z = 1 \).

8. (10 points) Compute the integral in the previous problem.