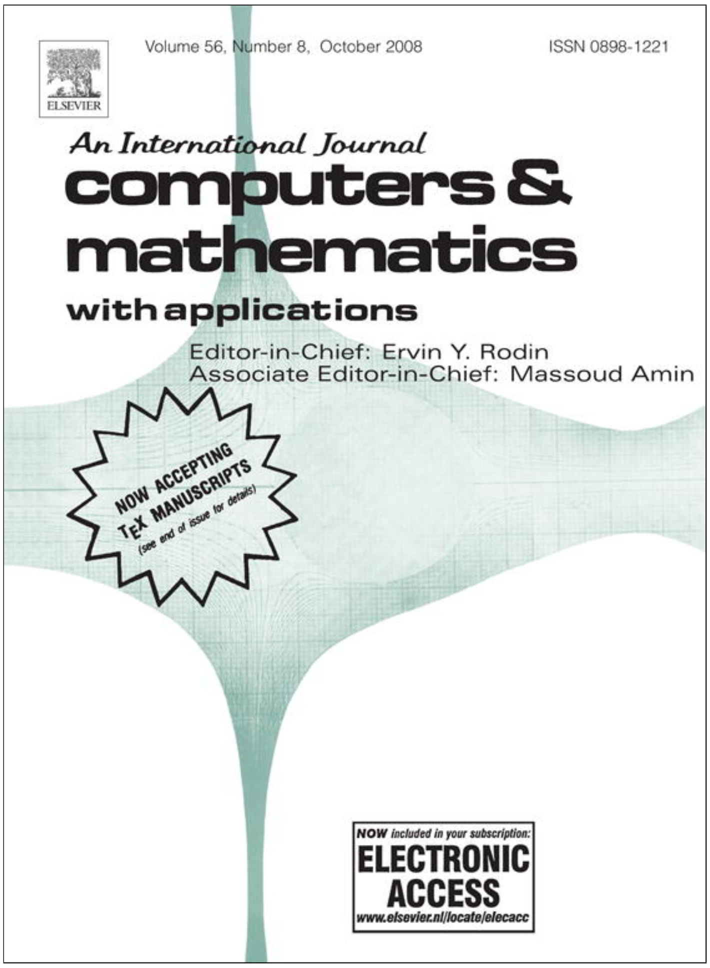
Volume 56, Number 8, October 2008
ISSN 0898-1221
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computers & mathematics
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Computers and Mathematics with Applications 56 (2008) 1975–1980

Contents lists available at ScienceDirect

Computers and Mathematics with Applications

journal homepage: www.elsevier.com/locate/camwa

Nyström methods and singular second-order differential equations

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ARTICLE INFO

Article history:
Received 28 November 2007
Received in revised form 3 April 2008
Accepted 17 April 2008

Keywords:
Numerical approximation
Singular differential equations
Nyström method
Lane–Emden equation
Second-order differential equation

ABSTRACT

Chawla, Jain and Subramanian studied the application of Nyström methods to a class of singular initial value problems. Following their approach, we generalize this class by applying the Nyström method to the initial value problem for an equation of the form $y'' + p(t)y' + q(t, y(t)) = 0,\ t \in (0, 1]$ where p has a certain specified type of singularity and q is sufficiently differentiable, and then we determine the order of convergence. This is followed by computational evidence.

1. Introduction

The solution of second-order initial value problems generally is accomplished by using one of two procedures. The first idea is to convert the IVP to a system of first-order equations and then apply a solution technique. The second method is to apply a direct technique without conversion to a system. The methods of Nyström are examples of this second idea. Nyström methods have been used to numerically approximate the solution to many initial value problems. See for example Refs. [1–13].

In [2] Chawla, Jain and Subramanian considered the application of Nyström methods to the initial value problem

$$y'' + \frac{2}{t}y' + f(t, y) = 0, \quad 0 < t < t_f,\ y(0) = a,\ y'(0) = 0.$$

In what follows we will generalize this class by making certain modifications of the proof to initial value problems of the form

$$\begin{aligned} &y'' + p(t)y' + q(t, y(t)) = 0, \quad t \in (0, 1] \\ &y(0) = a, \qquad y'(0) = b. \end{aligned} \tag{IVP1}$$

While the class in [2] does generalize certain well-known singular IVP's such as that involving the Lane–Emden equation, this new class allows for more complicated coefficients of the y' term. For more on the Lane–Emden equation see [14]. For applications of our results, we note that one can easily find examples of functions p different from $\frac{2}{t}$ that are covered by our theorem—see the discussion following the proof. Also consider the following: One can prove that any p of the form $\frac{2}{t} + g(t)$, where $g(0) = 0$ and g is continuously differentiable on $[0, 1]$, satisfies the hypotheses of our theorem. So, for example, our theorem could be used for perturbations increasing with time for any of the applications with $p(t) = \frac{2}{t}$ found in the

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doi:10.1016/j.camwa.2008.04.023

reference list. Also hypothesis A5 can be replaced with the hypotheses of any existence and uniqueness theorem such as the one found in [15], but making assumption A5 directly instead is strictly weaker.

We assume that p might have a singularity at the origin. In what follows $m(t) = e^{\frac{1}{2}\int_1^t p(s)\mathrm{d}s}$. We will assume the following conditions.

(A1) $p \geq 0$ on $(0, 1]$ and $p \in C^1((0, 1])$.
(A2) $2p'(t) + (p(t))^2 = k(t)$ on $(0, 1]$ for some $k \in C([0, 1])$.
(A3) $\frac{t}{m(t)}$ is bounded on $[0, 1]$, say by Υ.
(A4) q is continuous on $[0, 1]\times\mathbf{R}$ and $q(t, y)$ is differentiable sufficiently many times in the second variable for each $t \in [0, 1]$ and measurable in t.
(A5) (IVP1) has a unique solution $y(t)$.
(A6) $\lim_{t\downarrow 0} m'(t)$ is finite.

2. Global error for the Nyström method

The main result is the following.

Theorem. *Under conditions*(A1)–(A6), *the global error of the Nyström method as characterized in* [10] *with* $s - 1$ *function evaluations is* $O(h^s)$.

Proof. Let $\chi(t) = m(t)y(t)$ for $t > 0$, $\chi(0) = a\lim_{t\downarrow 0} m(t)$ and $\chi'(0) = b\lim_{t\downarrow 0} m(t) + a\lim_{t\downarrow 0} m'(t)$. Note that $m \in C[0, 1]$ and m is nondecreasing. Also, $m''/m \in C[0, 1]$ by A2. Hence, $m', m'' \in C[0.1]$. Using (IVP1) we find that $\chi''(t) = m''(t)y(t) + 2m'(t)y'(t) + m(t)(-p(t)y'(t) - q(t, y))$ and since $m'(t) = \frac{m(t)p(t)}{2}$ we have $\chi''(t) = m''(t)y(t) - m(t)q(t, y)$. Thus we have $\chi''(t) = m''(t)\frac{\chi(t)}{m(t)} - m(t)q(t, \frac{\chi(t)}{m(t)})$. We will let $F(t, \chi)$ represent the right-hand side and we have transformed our equation into $\chi''(t) = F(t, \chi)$.

Now we make the substitution $z = \chi'(t)$ and we let $t_n = nh$, $n = 0, 1, \ldots, N$ for $h = \frac{1}{N}$. We will let $z_n = z(t_n)$, $m(t_n) = m_n$, and $\chi_n = \chi(t_n)$. We are ready to apply Nyström's method.

$$\chi_{n+1} = \chi_n + hz_n + h^2\sum_{j=1}^{s-1} a_jK_j + t_n(h), \tag{1}$$
$$z_{n+1} = z_n + h\sum_{j=1}^{s-1} b_jK_j + t_n'(h)$$

where for each $n = 0, 1, \ldots, N - 1$, $K_i = F(t_n + \alpha_i h, \chi_n + \alpha_i hz_n + h^2\sum_{j=1}^{i-1}\beta_{ij}K_j)$ for $i = 1, 2, \ldots, s - 1$ and $t_n(h), t_n'(h) = O(h^{s+1})$. Here a_j, b_j, α_i and β_{ij} are global constants—for details see [2,4,10]. Note: t_n and $t_n(h)$ denote different things.

Let $y_n = y(t_n)$. Now since $\chi_n = m_ny_n$ for each n we can rewrite (1) as

$$m_{n+1}y_{n+1} = m_ny_n + hz_n + h^2\sum_{j=1}^{s-1} a_jK_j + t_n(h) \tag{2}$$
$$z_{n+1} = z_n + h\sum_{j=1}^{s-1} b_jK_j + t_n'(h).$$

Leaving out the local truncation errors $t_n(h)$ and $t_n'(h)$ and replacing y_n, z_n, and K_j by $\widetilde{y_n}$, $\widetilde{z_n}$, and $\widetilde{K_j}$ respectively we have the numerical method

$$m_{n+1}\widetilde{y_{n+1}} = m_n\widetilde{y_n} + h\widetilde{z_n} + h^2\sum_{j=1}^{s-1} a_j\widetilde{K_j} \tag{3}$$
$$\widetilde{z_{n+1}} = \widetilde{z_n} + h\sum_{j=1}^{s-1} b_j\widetilde{K_j}.$$

In order to examine convergence let us represent the errors at each step by $e_n = y_n - \widetilde{y_n}$ and $d_n = z_n - \widetilde{z_n}$ and subtract Eqs. (3) from Eqs. (2).

$$m_{n+1}e_{n+1} = m_ne_n + hd_n + h^2\sum_{j=1}^{s-1} a_j(K_j - \widetilde{K_j}) + t_n(h) \tag{4}$$
$$d_{n+1} = d_n + h\sum_{j=1}^{s-1} b_j(K_j - \widetilde{K_j}) + t_n'(h).$$

Claim 1. *For* $n = 0, 1, \ldots, N$,

$$m_{n+1}e_n - m_ne_n = h(1 + h^2\mu_n)d_n + h^2\lambda_ne_n + t_n(h)$$
$$d_{n+1} - d_n = h\sigma_ne_n + h^2\delta_nd_n + t_n'(h),$$

where μ_n, λ_n, σ_n *and* δ_n *are defined in* (6).

Proof of Claim 1. Note that, letting $\widetilde{\chi}_n = m_n\widetilde{y}_n$,

$$K_i - \widetilde{K}_i = \left[m''(nh+\alpha_i h)/m(nh+\alpha_i h)\right](\chi_n - \widetilde{\chi}_n + \alpha_i h(z_n - \widetilde{z}_n) + h^2\sum_{j=1}^{i-1}\beta_{ij}(K_j - \widetilde{K}_j))$$
$$- m(nh+\alpha_i h)\left[q\left(nh+\alpha_i h, \frac{\chi_n + \alpha_i h z_n + h^2\sum_{j=1}^{i-1}\beta_{ij}K_j}{m(nh+\alpha_i h)}\right) - q\left(nh+\alpha_i h, \frac{\widetilde{\chi}_n + \alpha_i h\widetilde{z}_n + h^2\sum_{j=1}^{i-1}\beta_{ij}\widetilde{K}_j}{m(nh+\alpha_i h)}\right)\right]$$
$$= \left[m''(nh+\alpha_i h)/m(nh+\alpha_i h)\right]\left(m_n e_n + \alpha_i h d_n + h^2\sum_{j=1}^{i-1}\beta_{ij}(K_j - \widetilde{K}_j)\right)$$
$$- m(nh+\alpha_i h)\frac{\partial q}{\partial y}(nh+\alpha_i h, \xi_i)\cdot\left[\frac{m_n e_n + \alpha_i h d_n + h^2\sum_{j=1}^{i-1}\beta_{ij}(K_j - \widetilde{K}_j)}{m(nh+\alpha_i h)}\right].$$

This last equality comes from applying the Mean Value Theorem to q in the variable y which yields ξ_i in the correct interval. Let $V_{ni} = \frac{1}{m(nh+\alpha_i h)}\cdot(m''(nh+\alpha_i h) - \frac{\partial q}{\partial y}(nh+\alpha_i h, \xi_i)m(nh+\alpha_i h))$. Then $K_i - \widetilde{K}_i = V_{ni}(m_n e_n + \alpha_i h d_n + h^2\sum_{j=1}^{i-1}\beta_{ij}(K_j - \widetilde{K}_j))$ and thus $K_i - \widetilde{K}_i - h^2\sum_{j=1}^{i-1}V_{ni}\beta_{ij}(K_j - \widetilde{K}_j) = V_{ni}(m_n e_n + \alpha_i h d_n)$ for $i = 1, 2, \ldots, s-1$. Let $\mathbf{I}$ be the $(s-1)\times(s-1)$ identity matrix, $\mathbf{V}_n = \mathrm{diag}(V_{n,1}, \ldots, V_{n,s-1})$, and let $\boldsymbol{\beta} = (\beta_{ij})$ be the strictly lower triangular matrix formed by using β_{im}, $m = 1, \ldots, i-1$, $i = 1, \ldots, s-1$. Now we let $\mathbf{1} = (1, 1, 1, \ldots, 1)^{\mathrm{T}}$, $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{s-1})^{\mathrm{T}}$, and $\mathbf{K} - \widetilde{\mathbf{K}} = (K_1 - \widetilde{K}_1, \ldots, K_{s-1} - \widetilde{K}_{s-1})^{\mathrm{T}}$. We then obtain

$$(\mathbf{I} - h^2\mathbf{V}_n\boldsymbol{\beta})(\mathbf{K} - \widetilde{\mathbf{K}}) = m_n e_n \mathbf{V}_n\mathbf{1} + h d_n \mathbf{V}_n\boldsymbol{\alpha}. \tag{5}$$

$\mathbf{I} - h^2\mathbf{V}_n\boldsymbol{\beta}$ has ones down the main diagonal and is lower triangular so it is nonsingular. Thus we set $\mathbf{M}_n = (\mathbf{I} - h^2\mathbf{V}_n\boldsymbol{\beta})^{-1}\mathbf{V}_n$. Let $\mathbf{a} = (a_1, \ldots, a_{s-1})^{\mathrm{T}}$, $\mathbf{b} = (b_1, \ldots, b_{s-1})^{\mathrm{T}}$ and substitute into (5) to obtain $\mathbf{K} - \widetilde{\mathbf{K}} = m_n e_n \mathbf{M}_n\mathbf{1} + h d_n \mathbf{M}_n\boldsymbol{\alpha}$. Therefore $\sum_{j=1}^{s-1}a_j(K_j - \widetilde{K}_j) = \mathbf{a}^{\mathrm{T}}\mathbf{M}_n(m_n e_n\mathbf{1} + h d_n\boldsymbol{\alpha})$. There is a similar expression for $\sum_{j=1}^{s-1}b_j(K_j - \widetilde{K}_j)$.

Let

$$\mu_n = \mathbf{a}^{\mathrm{T}}\mathbf{M}_n\boldsymbol{\alpha}, \qquad \lambda_n = m_n\mathbf{a}^{\mathrm{T}}\mathbf{M}_n\mathbf{1}, \qquad \sigma_n = m_n\mathbf{b}^{\mathrm{T}}\mathbf{M}_n\mathbf{1} \quad \text{and} \quad \delta_n = \mathbf{b}^{\mathrm{T}}\mathbf{M}_n\boldsymbol{\alpha}. \tag{6}$$

Now rewriting (4) we obtain

$$m_{n+1}e_{n+1} - m_n e_n = h d_n + h^2\mathbf{a}^{\mathrm{T}}\mathbf{M}_n(m_n e_n\mathbf{1} + h d_n\boldsymbol{\alpha}) + t_n(h)$$
$$d_{n+1} - d_n = h\mathbf{b}^{\mathrm{T}}\mathbf{M}_n(m_n e_n\mathbf{1} + h d_n\boldsymbol{\alpha}) + t'_n(h).$$

Thus we have

$$m_{n+1}e_n - m_n e_n = h(1 + h^2\mu_n)d_n + h^2\lambda_n e_n + t_n(h) \tag{7}$$
$$d_{n+1} - d_n = h\sigma_n e_n + h^2\delta_n d_n + t'_n(h),$$

proving Claim 1. ■

Now we introduce some $N+1$ dimensional vectors and $(N+1)\times(N+1)$ matrices. Let $\mathbf{e} = (e_0, e_1, \ldots, e_N)^{\mathrm{T}}$, $\mathbf{d} = (d_0, \ldots, d_N)^{\mathrm{T}}$, $\mathbf{T}(h) = (0, t_0(h), t_1(h), \ldots, t_{N-1}(h))^{\mathrm{T}}$ and $\mathbf{T}'(h) = (0, t'_0(h), t'_1(h), \ldots, t'_{N-1}(h))^{\mathrm{T}}$. Let $\mathbf{D} = (d_{ij})_{i,j=0}^{N}$ be the lower triangular matrix defined by $d_{00} = 1$, $d_{ii} = m_i$, $d_{ii-1} = -m_{i-1}$, $i = 1, \ldots, N$, $d_{ij} = 0$ for $j \le i-2$. Let $\mathbf{C}$ be the lower triangular $(N+1)\times(N+1)$ matrix formed by placing ones down the main diagonal, negative ones down the first subdiagonal, and zeros elsewhere. We also define $\boldsymbol{\lambda} = \mathrm{subdiag}\{\lambda_0, \lambda_1, \lambda_2, \ldots, \lambda_{N-1}\}$, $\boldsymbol{\mu} = \mathrm{subdiag}\{1+h^2\mu_0, 1+h^2\mu_1, \ldots, 1+h^2\mu_{N-1}\}$, $\boldsymbol{\sigma} = \mathrm{subdiag}\{\sigma_0, \sigma_1, \ldots, \sigma_{N-1}\}$ and $\boldsymbol{\delta} = \mathrm{subdiag}\{\delta_0, \delta_1, \ldots, \delta_{N-1}\}$, where the subdiag denotes a matrix with zeros everywhere except for the specified values on the first subdiagonal. Note that $\mathbf{C} - h^2\boldsymbol{\delta}$ is lower triangular with nonzero diagonal entries and is thus nonsingular. $\mathbf{D}$ has the same property since $m_n = \mathrm{e}^{\frac{1}{2}\int_1^{nh}p(u)\mathrm{d}u} \neq 0$ for $n > 0$.

We now let

$$\mathbf{P} = \mathbf{D}^{-1}[h\boldsymbol{\lambda} + h\boldsymbol{\mu}(\mathbf{C} - h^2\boldsymbol{\delta})^{-1}\boldsymbol{\sigma}]$$

and

$$\mathbf{r}(h) = \mathbf{D}^{-1}[h\boldsymbol{\mu}(\mathbf{C} - h^2\boldsymbol{\delta})^{-1}\mathbf{T}'(h) + \mathbf{T}(h)].$$

Claim 2.

$$\mathbf{e} = h\mathbf{P}\mathbf{e} + \mathbf{r}(h).$$

Proof of Claim 2. Rewriting (7) we find:

$$\mathbf{D}\mathbf{e} = h^2\boldsymbol{\lambda}\mathbf{e} + h\boldsymbol{\mu}\mathbf{d} + \mathbf{T}(h)$$
$$(\mathbf{C} - h^2\boldsymbol{\delta})\mathbf{d} = h\boldsymbol{\sigma}\mathbf{e} + \mathbf{T}'(h).$$

$$\mathbf{e} = \mathbf{D}^{-1}(h^2\boldsymbol{\lambda}\mathbf{e} + h\boldsymbol{\mu}\mathbf{d} + \mathbf{T}(h)) = h\mathbf{D}^{-1}[h\boldsymbol{\lambda}\mathbf{e} + \boldsymbol{\mu}\mathbf{d}] + \mathbf{D}^{-1}[\mathbf{T}(h)].$$

$$\begin{aligned}\therefore \mathbf{e} &= h\mathbf{D}^{-1}[h\boldsymbol{\lambda}\mathbf{e} + \boldsymbol{\mu}(\mathbf{C} - h^2\boldsymbol{\delta})^{-1}(h\boldsymbol{\sigma}\mathbf{e} + \mathbf{T}'(h))] + \mathbf{D}^{-1}[\mathbf{T}(h)] \\ &= h\mathbf{D}^{-1}[h\boldsymbol{\lambda} + h\boldsymbol{\mu}(\mathbf{C} - h^2\boldsymbol{\delta})^{-1}\boldsymbol{\sigma}]\mathbf{e} + h\mathbf{D}^{-1}[\boldsymbol{\mu}(\mathbf{C} - h^2\boldsymbol{\delta})^{-1}\mathbf{T}'(h)] + \mathbf{D}^{-1}[\mathbf{T}(h)]\end{aligned}$$

thus $\mathbf{e} = h\mathbf{P}\mathbf{e} + \mathbf{r}(h)$, proving Claim 2. ■

Claim 3. *$\max_{0\le i,j\le N} |P_{ij}|$ is bounded for sufficiently small h by a constant independent of N.*

Proof of Claim 3. **P** is strictly lower triangular because the product of a strictly lower triangular matrix and a lower triangular matrix is strictly lower triangular, and also $\mathbf{D}^{-1}$ is lower triangular since **D** is. Note that the lower triangular matrix $\mathbf{D}^{-1} = (d_{ij}^{-1})$ where $d_{00}^{-1} = 1$, $d_{i0}^{-1} = 0$, $i = 1, 2, \dots, N$, $d_{ij}^{-1} = \frac{1}{m_j}$, $i = 1, 2, \dots, N$, $j = 1, 2, \dots, i$. Let $\|\cdot\|$ denote the $\infty-$ norm of a vector or matrix. Note that $V_{ni} = \frac{1}{2}p'(nh + \alpha_i h) + \frac{p^2(nh+\alpha_i h)}{4} - \frac{\partial q}{\partial y}(nh + \alpha_i h, \xi_i)$. This is because $\frac{m''(nh+\alpha_i h)}{m(nh+\alpha_i h)} = \frac{1}{2}p'(nh + \alpha_i h) + \frac{p^2(nh+\alpha_i h)}{4}$. It can be shown that $\left|\frac{\partial q}{\partial y}(nh + \alpha_i h, \xi_i)\right|$ is bounded, say by L, for all n and i, and h sufficiently small. Thus $|V_{ni}| \le \frac{1}{4}B + L$ where $B = \sup_{t\in[0,1]} |k(t)|$. Define L_1 by $L_1 = \frac{1}{4}B + L$. Thus we have $\|\mathbf{V}_n\|_\infty \le L_1$, $n = 0, 1, \dots, N-1$. This bound is independent of h.

Recall that if A is a matrix with $\|A\| < 1$, then $I - A$ is invertible and $\|I - A\| \le \frac{1}{1-\|A\|}$. Thus, $\left(I - h^2V_n\beta\right)M_n = V_n$ implies

$$\|M_n\| \le \left\|\left(I - h^2V_n\beta\right)^{-1}\right\| \|V_n\| \le \frac{L_1}{1 - h^2L_1c_\beta} \tag{8}$$

where $c_\beta = \|\boldsymbol{\beta}\|$ and h is sufficiently small so that $h^2L_1c_\beta < 1$. In view of (8) clearly we can find c_μ and c_δ such that $|\delta_n| \le c_\delta$ and $|1 + h^2\mu_n| \le c_\mu$ for all n. Since $m(t)$ is bounded from [15] we can also find c_λ and c_σ such that $|\lambda_n| \le c_\lambda$ and $|\sigma_n| \le c_\sigma$ for every n. Note that these four constants are independent of N. Note that

$$\therefore \max_{0\le i,j\le N} |\mathbf{D}^{-1}\boldsymbol{\lambda}| \le \max_{i\ge j+1, j=0,1,\dots,N-1} \frac{|\lambda_j|}{m_i} \le \frac{c_\lambda}{m_1}.$$

This is because $m(t)$ in [15] is shown to be nondecreasing. Let $\mathbf{C}^{-1} = (c_{ij}^{-1})$ be the inverse of **C.** We find that $c_{ij}^{-1} = 1$ for $i = 1, 2, \dots, N$, $j = 1, 2, \dots, i$. Therefore $\|\mathbf{C}^{-1}\boldsymbol{\delta}\| \le Nc_\delta$. Also $\max_{0\le i,j\le N} |(\mathbf{D}^{-1}\boldsymbol{\mu}(C - h^2\boldsymbol{\delta})^{-1}\sigma)_{ij}| = \max_{0\le i,j\le N} |(\mathbf{D}^{-1}\boldsymbol{\mu}(I - h^2\mathbf{C}^{-1}\boldsymbol{\delta})^{-1}\mathbf{C}^{-1}\sigma)_{ij}|$.

Now we turn our attention to **P**.

$$\begin{aligned}\max_{0\le i,j\le N} \left|P_{ij}\right| &= \max_{0\le i,j\le N} \left|\mathbf{D}^{-1}\left[h\boldsymbol{\lambda} + h\boldsymbol{\mu}(C - h^2\boldsymbol{\delta})^{-1}\boldsymbol{\sigma}\right]_{ij}\right| \\ &\le \frac{c_\lambda h}{m_1} + \max_{0\le i,j\le N} \left|\mathbf{D}^{-1}\left[h\boldsymbol{\mu}(\mathbf{I} - h^2\mathbf{C}^{-1}\boldsymbol{\delta})^{-1}\mathbf{C}^{-1}\boldsymbol{\sigma}\right]_{ij}\right|.\end{aligned} \tag{9}$$

Here, $\frac{c_\lambda h}{m_1}$ is bounded by $c_\lambda\Upsilon$ by assumption A3. To finish the proof of Claim 3, we now show that the maximum in (9) is bounded. Let $W = h^2C^{-1}\boldsymbol{\delta}$. Observe that W is a strictly lower triangular matrix and each entry in absolute value is less than or equal to $c_\delta h^2$, so $\|W\| < 1$ if h is small. W^k is also strictly lower triangular for any $k \ge 1$. Now we use the Neumann series expansion

$$(I - W)^{-1} = I + W + W^2 + \cdots.$$

(This in fact is a finite series.) If A and B are matrices of size $(N+1)\times(N+1)$ whose entries in absolute value are bounded by a and b respectively, then the entries of AB are bounded in absolute value by $(N+1)ab$. Using this and mathematical induction, we get that the entries of W^k are bounded in absolute value by $c_\delta^k(N+1)^{k-1}h^{2k}$ for any $k \ge 1$. So each entry of $W + W^2 + \cdots$ is bounded in absolute value by

$$\sum_{k=1}^{\infty} c_\delta^k(N+1)^{k-1}h^{2k} = \frac{c_\delta h^2}{1 - c_\delta(N+1)h^2} \le C_1h^2,$$

assuming h is small (since $(N+1)h$ is bounded). Hence the entries of $(I - W)^{-1}$ are bounded in absolute value by C_1h^2, except the main diagonal where we have ones. $C^{-1}\boldsymbol{\sigma}$ is a strictly lower triangular matrix whose entries are bounded by c_σ in absolute value. It follows that the entries of $(I - W)^{-1}C^{-1}\boldsymbol{\sigma}$ are bounded by $c_\sigma(1 + NCh^2) \le C_2$ in absolute value, where C_2 is a global constant. Let $d_{ij}^{\boldsymbol{\mu}}$ denote the entries of $D^{-1}\boldsymbol{\mu}$. Using the expressions we have for the entries d_{ij} of D^{-1}, we obtain $\left|d_{ij}^{\boldsymbol{\mu}}\right| \le c_\mu\frac{1}{m_i}$, $i = 0, 1, \dots, N$, $j = 0, 1, \dots, i-1$ and $d_{ij}^{\boldsymbol{\mu}} = 0$ otherwise. Let D_2 denote the matrix with entries $d_{ij}^{(2)} = \frac{1}{m_i}$, $i = 0, 1, \dots, N$, $j = 0, 1, \dots, i-1$ and $d_{ij}^{(2)} = 0$ otherwise. Based on the estimates above, to get a bound of the entries of

$$hD^{-1}\boldsymbol{\mu}(I - W)^{-1}C^{-1}\sigma \tag{10}$$

we must calculate $hD_2C_2I^*$, where I^* is the full lower triangular matrix, i.e., a matrix with 0's down the main diagonal and above the main diagonal, and 1's below the main diagonal. The largest entries of hD_2I^* in each row are in the first column. They are 0, 0, $\frac{1}{m_2}h$, $\frac{2}{m_3}h, \ldots, \frac{N-1}{m_N}h$. These are bounded by $\max_{2\le k\le N}\frac{kh}{m_k} \le \Upsilon$ by A3. So, the entries of (10) are bounded by $C_2\Upsilon$, completing the proof of Claim 3. ■

Claim 4. $\|\mathbf{r}(h)\| \le \|\mathbf{D}^{-1}\mathbf{T}(h)\| + \Upsilon\frac{c_\mu}{1-hc_\delta}\|\mathbf{C}^{-1}\mathbf{T}'(h)\|$.

Proof of Claim 4. We have

$$\begin{aligned}\|\mathbf{r}(h)\| &= \|\mathbf{D}^{-1}[h\boldsymbol{\mu}(\mathbf{C}-h^2\boldsymbol{\delta})^{-1}\mathbf{T}'(h)+\mathbf{T}(h)]\| \le \|\mathbf{D}^{-1}\mathbf{T}(h)\| + \Upsilon c_\mu\|(\mathbf{C}-h^2\boldsymbol{\delta})^{-1}\mathbf{T}'(h)\|\\ &= \|\mathbf{D}^{-1}\mathbf{T}(h)\| + \Upsilon c_\mu\|(\mathbf{I}-h^2\mathbf{C}^{-1}\boldsymbol{\delta})^{-1}\mathbf{C}^{-1}\mathbf{T}'(h)\|\\ &\le \|\mathbf{D}^{-1}\mathbf{T}(h)\| + \Upsilon\frac{c_\mu}{1-hc_\delta}\|\mathbf{C}^{-1}\mathbf{T}'(h)\|.\end{aligned}$$

Claim 4 is thus established. ■

Now let us determine the order of convergence.

Claim 5. $\|\mathbf{r}(h)\| = O(h^s)$.

Proof of Claim 5. Observe that $\|\mathbf{T}(h)\| = O(h^{s+1})$ and $\|\mathbf{C}^{-1}\mathbf{T}'(h)\| \le NM'h^{s+1} = M'h^s$, where M' is the constant in the definition of $O(h^{s+1})$. Similarly, $\|\mathbf{D}^{-1}\mathbf{T}(h)\| = \max_{1\le k\le N}|\frac{1}{m_k}\sum_{i=0}^{k-1}t_i(h)| \le \max_{1\le k\le N}\frac{1}{m_k}kMh^{s+1}$ for some positive constant M. Since $\left|\frac{kh}{m(kh)}\right| \le \Upsilon$ by assumption (A3), it follows that $\|\mathbf{D}^{-1}\mathbf{T}(h)\| \le \Upsilon Mh^s$ and hence is $O(h^s)$. Applying these results to the result of Claim 4 shows that $\|\mathbf{r}(h)\| = O(h^s)$, establishing Claim 5. ■

Recall that in Claim 2 we showed that $\mathbf{e} = h\mathbf{Pe} + \mathbf{r}(h)$, therefore $|e_i| \le |r_i(h)| + h\max_{0\le i,j\le N}|P_{ij}|\sum_{j=0}^{i-1}|e_j| \le K'h^s + B'h\sum_{j=0}^{i-1}|e_j|$ by Claims 3 and 5 for some $B', K' > 0$, for all $i = 0, 1, \ldots, N$. Recalling that $ih \le 1$ and applying the discrete version of Gronwall's lemma yields

$|e_i| \le K'h^s e^{B'ih} \le K'e^{B'}h^s$ for all $i = 0, 1, \ldots, N$, i.e., $|e_i|$ is $O(h^s)$. ■

Thus the Nyström method yields the proper order of convergence. Note that hypothesis A2 in the theorem concerns a type of Riccati equation. These have been extensively studied so that there are many $p's$ to which this theorem can be applied. Of course, in an application, p would be given, so that one would not solve the differential equation in A2, but simply check to see if the given p satisfies A2. All of these hypotheses allow well-known special cases such as $p(t) = \frac{2}{t}$ as in [2] as well as some additional choices. For example it is easy to see that $p(t) = \cot(\frac{t}{2})$ and $p(t) = \frac{k+k\exp(-kt)}{1-\exp(-kt)}$ for $k \ne 0$ satisfy the required conditions.

3. Numerical results

Example 3.1. We provide some numerical results for the Nyström method presented in the previous section. We consider the initial value problem

$$y'' + \cot\left(\frac{t}{2}\right)y' - 1 = 0, \quad t \in (0, 1]$$
$$y(0) = 1, \qquad y'(0) = 0.$$

This problem has solution $u(t) = 3 - t\cot\left(\frac{t}{2}\right)$ for $t \in (0, 1]$. We apply the Nyström method with $s = 2$ (in which $a_1 = \frac{1}{2}$, $b_1 = 1$, and $\alpha_1 = \frac{1}{2}$) to obtain an approximate solution. Comparison of the approximate solution with the actual solution for several values of the step size h yields the following results, which exhibit the predicted $O(h^2)$ convergence.

t	Error for $h = .1$	Error for $h = .05$	Error for $h = .025$	Approximation for $h = .025$
0	0	0	0	1
0.1	0.000625365	0.000156289	0.000039069	1.00171
0.2	0.000625625	0.000156354	0.0000390853	1.00671
0.3	0.000626057	0.000156462	0.0000391123	1.01506
0.4	0.000626659	0.000156613	0.0000391499	1.02678
0.5	0.00062743	0.000156805	0.0000391981	1.04188
0.6	0.000628365	0.000157039	0.0000392565	1.0604
0.7	0.000629461	0.000157313	0.000039325	1.08238
0.8	0.000630713	0.000157626	0.0000394033	1.10786
0.9	0.000632116	0.000157977	0.0000394909	1.1369
1.0	0.000633662	0.000158363	0.0000395876	1.16955

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