

## The support of the equilibrium measure

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**Abstract** Let  $w(x) := \exp(-Q(x))$  be a non-negative weight function on the closed set  $\Sigma \subset \mathbb{R}$ . We will introduce some conditions, all of which guarantees that the support of the equilibrium measure associated with  $w$  is an interval. For example: if  $\exp(Q(x))$  is convex, then the support of the equilibrium measure is an interval. These type of conditions are very useful when determining the equilibrium measure. An integral representation formula for the equilibrium measure is also given.

Potential theory connects many areas and problems of mathematics. Examples include the electrostatic equilibrium, the weighted transfinite diameter, weighted Chebyshev polynomials, zero distribution of orthogonal polynomials, fast decreasing polynomials, incomplete polynomials, approximation by weighted polynomials, Padé approximation, Hankel determinants, and random matrices ([4], [9]).

Let  $w(x)$  be an *admissible* weight function defined on a closed set  $\Sigma \subset \mathbb{R}$  (precise definitions will be given below). The basic problem in weighted potential theory is to find the unit measure - called the *equilibrium measure* associated with  $w$  - which minimizes the *weighted energy integral*:

$$I_w(\mu) := - \int \int \log(|x - y|w(x)w(y))d\mu(x)d\mu(y).$$

The function  $Q(x) := \log(1/w(x))$  is called the *external field*.

The determination of the equilibrium measure is usually a difficult or impossible task. One should first find the support  $S_w$  of the equilibrium measure. If we can show that the support consists of  $N$  intervals, then theoretically we can use the method described in [5] to find the location of

these intervals. Then, at least theoretically, the equilibrium measure may be obtained from the Riemann-Hilbert problem.

This is the reason why it is important to have conditions which guarantee that the support is an interval, or the union of several intervals. These type of conditions are also used as assumptions on the external field at many problems. For example, the convexity of  $Q(x)$  was assumed in the formulation of the Borwein-Saff conjecture. This was a problem of approximation by weighted polynomials of the form  $w^n(x)P_n(x)$ , proven by Totik ([10]).

There are only a few conditions in the literature which guarantee that the support of the equilibrium measure is a single interval. The following two conditions are well known ([8]):

- 1.) If  $Q$  is convex on the interval  $I$ , then  $S_w \cap I$  is an interval.
- 2.) If  $\Sigma := [0, +\infty)$  and  $xQ'(x)$  is increasing on the interval  $I \subset \Sigma$  then  $S_w \cap I$  is an interval.

A condition similar to 2.) is introduced in [1], which is weaker than both 1.) and 2.). In this paper some other conditions are given. Their combination (which we will call *weak convexity condition*) is weaker than the ones given in [1]. Other type of conditions can be found in [2], [3], [6] and [7], which guarantee that the support of the equilibrium measure is an interval, or the union of several intervals. In [4] Deift, Kriecherbauer and McLaughlin showed that for a real analytic external field the support always consists of a finite number of intervals.

The reader can find the definition of the logarithmic capacity in [9], I.1. We say that a property holds quasi-everywhere, if the set where it does not hold has capacity 0.

Let  $\Sigma \subset \mathbb{R}$  be a closed set. A non-negative weight function  $w$  on  $\Sigma$  is said to be *admissible*, if it satisfies the following three conditions:

- (i)  $w$  is upper semi-continuous,
- (ii)  $\Sigma_0 := \{x \in \Sigma : w(x) > 0\}$  has positive capacity,
- (iii) if  $\Sigma$  is unbounded, then  $|z|w(z) \rightarrow 0$  as  $|z| \rightarrow \infty$ ,  $z \in \Sigma$ .

We define  $Q$  by

$$w(x) =: \exp(-Q(x)),$$

so  $Q : \Sigma \rightarrow (-\infty, \infty]$  is a lower semi-continuous function. This  $Q$  is called the external field.

Let  $\mathcal{M}(\Sigma)$  be the collection of all positive unit Borel measures with com-

compact support in  $\Sigma$ . We define the *logarithmic potential* of  $\mu \in \mathcal{M}(\Sigma)$  as

$$U^\mu(x) := \int \log \frac{1}{|x-t|} d\mu(t),$$

and the *weighted energy integral* as

$$I_w(\mu) := - \int \int \log(|x-y|w(x)w(y)) d\mu(x)d\mu(y).$$

We will need the following basic theorem ([9], Theorem I.1.3):

**Theorem A** *Let  $w$  be an admissible weight on the closed set  $\Sigma$  and let  $V_w := \inf\{I_w(\mu) : \mu \in \mathcal{M}(\Sigma)\}$ . Then*

- (a)  $V_w$  is finite,
- (b) there exists a unique element  $\mu_w \in \mathcal{M}(\Sigma)$  such that  $I_w(\mu_w) = V_w$ ,
- (c) setting  $F_w := V_w - \int Q d\mu_w$ , the inequality

$$U^{\mu_w}(x) + Q(x) \geq F_w$$

holds quasi-everywhere on  $\Sigma$ ,

- (d) the inequality

$$U^{\mu_w}(x) + Q(x) \leq F_w \tag{1}$$

holds for all  $x \in S_w := \text{supp}(\mu_w)$ .

**Remark** According to our definition, every measure in  $\mathcal{M}(\Sigma)$  has compact support, so the support  $S_w$  is a compact set.

The measure  $\mu_w$  is called *the equilibrium* or *extremal measure* associated with  $w$ .

**Notation 1** *Unless otherwise noted, all intervals in the paper are arbitrary intervals (that is, they may be open, half open, closed, bounded or unbounded). The notation  $I = [(a, b)]$  will mean that  $I$  is an interval (of any type) with endpoints  $a$  and  $b$ . The notation  $\text{int}(I)$  is used for the interior points of  $I$ . Absolute continuity inside  $I$  means that the function is absolutely continuous on any compact set which is lying in  $I$ .*

**Notation 2** *Throughout the article we agree on the following. Suppose that a function  $g(x)$  - which is usually defined by using  $Q'(x)$  - is said to be increasing (or decreasing) on a set  $E$ , but it is defined on a set  $F \subset E$ , where  $E \setminus F$  has measure zero. Then by " $f(x)$  is increasing on  $E$ " we mean*

the following: there exists a set  $G \subset F$  so that  $E \setminus G$  has measure zero and  $f(x)$  is increasing on  $G$ , i.e.,

$$f(x) \leq f(y), \quad \text{if } x < y, \quad x, y \in G. \quad (2)$$

In other words, we do not require that  $f(x)$  is increasing everywhere, where it is defined. (We will use the terminology "strictly increasing", if strict inequality is required in (2).)

Similarly, when we write that, say,  $1/(x - A) \leq Q'(x)$  on an interval  $(v, b)$ , we will mean that there exists a set  $G \subset (v, b)$  so that  $(v, b) \setminus G$  has measure 0,  $Q'(x)$  exists on  $G$  and  $1/(x - A) \leq Q'(x)$  on  $G$ .

These agreements weaken the assumptions on the functions in the theorems, but the given proofs are correct for this modified increasing/decreasing definition as well. When reading the proofs the reader should keep in mind that we are not working on the whole interval but on a subset of the interval having full measure, even though it is not indicated.

**Notation 3** We will say that a function  $g(x)$  is convex on an interval  $I$ , if  
a)  $g(x)$  is absolutely continuous inside  $I$ , (so  $g'$  exists a.e. in  $I$ ), and  
b)  $g'(x)$  is increasing on  $I$ . (By our agreement (see above) this means that there exists a set  $G$  such that  $I \setminus G$  has measure zero,  $g'(x)$  exists on  $G$ , and  $g'(x)$  is increasing on  $G$ .)

**Lemma 4** Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and let  $I \subset \Sigma_0$  be an interval. Let  $\mu_w$  be the equilibrium measure associated with  $w$ . Assume that  $Q$  is absolutely continuous inside  $I$  and that

$$\liminf_{\substack{x \rightarrow z \\ x \in I}} Q(x) = Q(z), \quad \text{if } z \text{ is an endpoint of } I \text{ with } z \in I. \quad (3)$$

If for some function  $f : \text{int}(I) \rightarrow \mathbb{R}^+$ , the function  $f(x) \frac{d}{dx} [U^{\mu_w}(x) + Q(x)]$  is increasing, but not constant function on any non-degenerated subinterval of  $\text{int}(I)$ , then  $S_w \cap I$  is an interval.

**Proof.** Suppose on the contrary that there exist  $a, b \in S_w \cap I$ ,  $a < b$ :  $(a, b) \cap S_w = \emptyset$ .

Let  $\mu := \mu_w$  denote the equilibrium measure associated with  $w$  and

$$U(x) := U^\mu(x) := \int_{\mathbb{R}} \ln \frac{1}{|x - t|} d\mu(t)$$

be the logarithmic potential function of  $\mu$ . Since  $U(x)$  is a  $C^\infty$  function on  $(a, b)$ , it is absolutely continuous on every closed subset of  $(a, b)$ . Because of the Lebesgue monotone convergence theorem,  $U(x)$  is continuous on  $[a, b]$ . (Indeed, we may assume that  $\text{diam}(S_w) \leq 1$ , so  $\ln(1/|x-t|) > 0 \quad x, t \in S_w$ . We split the above integral to two integrals, one with measure  $\mu|_{(-\infty, a]}$  and the other with measure  $\mu|_{[b, +\infty)}$ . Since  $U(a)$  and  $U(b)$  are finite from (1), we can apply Lebesgue's theorem to the two integrals.) So  $U(x)$  is absolutely continuous on  $[a, b]$ .

Let  $R(x) := U(x) + Q(x)$ . By our assumption  $f(x)R'(x)$  is increasing on  $I$ , i.e., there exists a set  $G \subset I$  so that  $I \setminus G$  has measure zero,  $f(x)R'(x)$  exists for all  $x \in G$ , and  $f(x)R'(x)$  is increasing on  $G$ . (See our agreement at Notation 2.)

It follows that we cannot find numbers  $x_1, x_2 \in \text{int}(I) \cap G$ ,  $x_1 < x_2$ , for which both  $0 < R'(x_1)$  and  $0 > R'(x_2)$  hold.

From Theorem A we have  $R(x) := U(x) + Q(x) \geq F_w \quad x \in (a, b)$  and  $R(a), R(b) \leq F_w$ . It is impossible that  $R(x) = F_w$  for all  $y \in (a, b) \cap G$ , because then  $f(x)R'(x)$  would be constant function on  $(a, b)$ . So there is a  $y \in (a, b) \cap G : F_w < R(y)$ . We have:

$$0 < R(y) - F_w \leq R(y) - R(a) = R(y) - \liminf_{x \rightarrow a+0} R(x) = \limsup_{x \rightarrow a+0} \int_x^y R'(t) dt$$

which implies the existence of  $x_1 \in (a, y) \cap G : 0 < R'(x_1)$ . Similarly

$$0 > F_w - R(y) \geq R(b) - R(y) = \liminf_{x \rightarrow b-0} R(x) - R(y) = \liminf_{x \rightarrow b-0} \int_y^x R'(t) dt,$$

so there is an  $x_2 \in (y, b) \cap G : 0 > R'(x_2)$ . This is a contradiction. ■

If  $Q$  is convex, then  $\exp(Q)$  is also convex, so the following theorem is a generalization of the well known theorem in which the convexity of  $Q$  is assumed.

**Theorem 5** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and suppose that  $\exp(Q)$  is convex on the interval  $I \subset \Sigma_0$  and satisfies condition (3). Then  $S_w \cap I$  is an interval.*

**Proof.** Suppose on the contrary that there exist  $a, b \in S_w \cap I$ ,  $a < b$ :  $(a, b) \cap S_w = \emptyset$ .

Let

$$m(x, y, t) := \frac{f(y)}{t-y} + f(y)Q'(y) - \frac{f(x)}{t-x} - f(x)Q'(x) \quad (4)$$

Our goal is to find a non-negative function  $f(x)$  such that for any fixed  $x < y$ , where  $x, y \in (a, b)$  we have:

$$0 \leq m(x, y, t) \quad t \in \mathbb{R} \setminus (a, b) \quad (5)$$

and equality holds for at most finitely many values of  $t$ .

If we can find such a function  $f(x)$ , then integrating (5) with respect to  $d\mu(t)$  gives

$$0 < f(y) \frac{d}{dy} [U^\mu(y) + Q(y)] - f(x) \frac{d}{dx} [U^\mu(x) + Q(x)] \quad (6)$$

(We have strict inequality, since  $\mu$  has measure zero on any finite set.) With Lemma 4 this will prove Theorem 5.

For fixed  $a < x < y < b$  let us find the infimum of  $t \rightarrow m(x, y, t)$  where  $t \in \mathbb{R} \setminus (a, b)$ . If  $0 < f(x)$  and  $0 < f(y)$ , but  $f(x) \neq f(y)$ , then the function  $t \rightarrow m(x, y, t)$  has exactly one critical point in  $\mathbb{R} \setminus [x, y]$ , which is:

$$t_0 := \frac{y\sqrt{f(x)} - x\sqrt{f(y)}}{\sqrt{f(x)} - \sqrt{f(y)}}, \quad (7)$$

and we have

$$m(x, y, t_0) = f(y)Q'(y) - f(x)Q'(x) - \frac{(\sqrt{f(y)} - \sqrt{f(x)})^2}{y-x} =: I(x, y). \quad (8)$$

Simple calculus argument shows that on  $\mathbb{R} \setminus (x, y)$  the one variable function  $t \mapsto m(x, y, t)$  achieves its minimum at  $t_0$  and nowhere else. (Indeed,  $\lim_{t \rightarrow x^-} m(x, y, t) = \lim_{t \rightarrow y^+} m(x, y, t) = +\infty$  and  $\lim_{t \rightarrow -\infty} m(x, y, t) = \lim_{t \rightarrow +\infty} m(x, y, t) = f(y)Q'(y) - f(x)Q'(x) > I(x, y)$ , so there must be a unique global minimum at the critical point  $t_0$ .)

If either  $f(x)$  or  $f(y)$  equals zero, but not both of them zero, then on  $\mathbb{R} \setminus (x, y)$  we still have a unique minimum at  $t_0$ , which is  $I(x, y)$  (though  $t_0$  is no longer a critical point).

Finally, if  $f(x) = f(y) > 0$ , then for all  $t \in \mathbb{R} \setminus (a, b)$  we have

$$I(x, y) = f(y)Q'(y) - f(x)Q'(x) < m(x, y, t) \quad (9)$$

In conclusion,

$$I(x, y) \leq m(x, y, t) \tag{10}$$

holds for all  $t \in \mathbb{R} \setminus (a, b)$ , and assuming not both  $f(x)$  and  $f(y)$  are zero, then equality holds for at most one value of  $t$ .

To finish the proof we claim that

$$f(x) := e^{2Q(x)}$$

satisfies  $0 \leq I(x, y)$ . Notice that if  $f(x)$  is defined this way, then  $f(x)Q'(x) = e^{2Q(x)}Q'(x) = h(x)h'(x)$  where  $h(x) := \sqrt{f(x)} = e^{Q(x)}$ . By our assumption  $h(x)$  is a convex function. We have

$$I(x, y) = h(y)h'(y) - h(x)h'(x) - \frac{(h(y) - h(x))^2}{y - x}.$$

Assume  $h(x) \leq h(y)$ . Using  $h'(x) \leq h'(y)$  we get

$$I(x, y) \geq (h(y) - h(x)) \left( h'(y) - \frac{h(y) - h(x)}{y - x} \right).$$

Here the first factor is non-negative and the second factor is also non-negative, since  $h(x)$  is a convex function.

Assume now  $h(x) > h(y)$ . Using  $h'(x) \leq h'(y)$  we get

$$I(x, y) \geq (h(y) - h(x)) \left( h'(x) - \frac{h(y) - h(x)}{y - x} \right).$$

Here the first factor is negative and the second factor is non-positive, since  $h(x)$  is a convex function.

So in all cases  $0 \leq I(x, y)$  and this together with (10) implies statement (5). ■

The following statement follows from Theorem 5 since Proposition 11 implies that  $\exp(Q)$  is convex. Nevertheless, we give another proof.

**Theorem 6** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and let  $I \subset \Sigma_0$  be an interval. Suppose that  $Q$  is absolutely continuous inside  $I$ , satisfies condition (3) and  $(x - A)^2Q'(x) - x$  is increasing on the interval  $I$ , where  $A \in \mathbb{R} \setminus \text{int}(I)$  is a fixed real number. Then  $S_w \cap I$  is an interval.*

**Proof.** As before, we would like to find a non-negative function  $f(x)$  such that (5) holds and we have strict inequality in (5) with the exception of finitely many values of  $t$ .

As in the proof of Theorem 5 it is enough to show that  $0 \leq I(x, y)$ . This time let us define  $f(x)$  as follows:

$$f(x) := (x - A)^2 \quad (11)$$

With this choice we have

$$\begin{aligned} I(x, y) &= f(y)Q'(y) - f(x)Q'(x) - \frac{(\sqrt{f(y)} - \sqrt{f(x)})^2}{y - x} \\ &\geq [(y - A)^2Q'(y) - y] - [(x - A)^2Q'(x) - x] \geq 0. \end{aligned} \quad (12)$$

since  $(x - A)^2Q'(x) - x$  is an increasing function by our assumption. ■

**Theorem 7** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and let  $I \subset \Sigma_0$  be an interval. Suppose that  $Q$  is absolutely continuous inside  $I$  and satisfies condition (3). Suppose that with some finite constants  $A < B$  we have  $I \subset [A, B]$ ,  $S_w \subset [A, B]$  and  $(x - A)(B - x)Q'(x) + x$  is increasing on the interval  $I$ . Then  $S_w \cap I$  is an interval.*

**Proof.** As before, we would like to find a non-negative function  $f(x)$  such that (5) holds and we have strict inequality in (5) with the exception of finitely many values of  $t$ .

Let us define  $f(x)$  as follows:

$$f(x) := (x - A)(B - x). \quad (13)$$

Let us assume first that  $f(x) \neq f(y)$ . We now show that  $t_0 \notin (A, B)$  holds ( $t_0$  is defined at (7)).

Assume that  $f(x) > f(y)$ . Now  $t_0 \leq A$  if and only if  $\sqrt{f(x)}(y - A) \leq \sqrt{f(y)}(x - A)$  which holds if and only if

$$(B - x)(y - A) \leq (B - y)(x - A). \quad (14)$$

Similarly,  $t_0 \geq B$  if and only if

$$(B - x)(y - A) \geq (B - y)(x - A). \quad (15)$$

Either (14) or (15) holds, which proves that  $t_0 \notin (A, B)$ . The proof of this observation is similar in the case when  $f(x) < f(y)$ .

In the case  $f(x) = f(y)$ ,  $t_0$  does not exist.

For fixed  $a < x < y < b$  we want to find the infimum of  $m(x, y, t)$  (see (4)) where  $t \in S_w$ . Since  $S_w \subset [A, B]$  and  $(a, b) \cap S_w = \emptyset$ , we have  $t \in [A, B] \setminus [x, y]$ . Using  $t_0 \notin (A, B)$  and calculus we gain that the minimum of  $m(x, y, t)$  is achieved at either  $t := A$  or  $t := B$  and nowhere else. It turns out that  $m(x, y, A)$  and  $m(x, y, B)$  has the same value which is

$$m(x, y, A) = m(x, y, B) = [(y-A)(B-y)Q'(y)+y]-[(x-A)(B-x)Q'(x)+x].$$

And this expression is non-negative, because of the assumption of the theorem.

Therefore  $0 \leq m(x, y, t)$  holds and we have strict inequality with the exception of at most two values of  $t$  ( $t := A$  and  $t := B$ ). This proves Theorem (7). ■

**Theorem 8** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and let  $I \subset \Sigma_0$  be an interval. Suppose that  $Q$  is absolutely continuous inside  $I$  and satisfies condition (3). Suppose that with some constant  $A \in \mathbb{R}$  we have  $I \subset [A, +\infty)$ ,  $S_w \subset [A, +\infty)$  and  $(x - A)Q'(x)$  is increasing on the interval  $I$ . Then  $S_w \cap I$  is an interval.*

**Proof.** The proof of this theorem is identical to the proof of Theorem 7, with the exception that we define  $f(x) := x - A$ . Exactly the same way as in the proof of Theorem 7, we can see that if  $f(x) \neq f(y)$ , then  $t_0 \notin (A, +\infty)$ . We also see the same way that the minimum of  $m(x, y, t)$ ,  $t \in [A, +\infty) \setminus [x, y]$  is achieved at either  $A$  or when  $t \rightarrow +\infty$ . It turns out that

$$m(x, y, A) = \lim_{t \rightarrow +\infty} m(x, y, t) = (y - A)Q'(y) - (x - A)Q'(x),$$

which expression is non-negative. ■

The symmetric version of Theorem 8 also holds. That version can be used when for some constant  $B \in \mathbb{R}$  we have  $S_w \subset (-\infty, B)$  and  $(B - x)Q'(x)$  is increasing on  $I$ .

In the following definition we introduce the "weak convexity" terminology. This is not the same as what was in [1]. The definition which follows is much more general.

**Definition 9** We say that a function  $Q : D \rightarrow \overline{\mathbb{R}}$  ( $D \subset \mathbb{R}$ ) is weak convex on an interval  $I = [(a, b)] \subset D$  ( $a, b \in \overline{\mathbb{R}}$ ) with basepoints  $A, B \in \overline{\mathbb{R}}$ ,  $A < B$ , if the following properties hold:

- (i)  $I \subset [A, B]$ ,
- (ii)  $Q$  is absolutely continuous inside  $(a, b)$  (so  $Q'$  exists a.e. in  $I$ ),
- (iii) if  $a \in I$ , then

$$\liminf_{x \rightarrow a+0} Q(x) = Q(a),$$

and if  $b \in I$ , then

$$\liminf_{x \rightarrow b-0} Q(x) = Q(b),$$

(iv)  $I$  can be written as the disjoint union of finitely many intervals  $I_1, \dots, I_n$  such that for any interval  $I_k$  ( $1 \leq k \leq n$ ):

$$\exp(Q(x)) \quad \text{is convex on } I_k, \quad \text{or} \quad (16)$$

$$(x - A)(B - x)Q'(x) + x \quad \text{is increasing on } I_k. \quad (17)$$

If  $-\infty < A$  and  $B = +\infty$  then (17) should be replaced by:

$$(x - A)Q'(x) \quad \text{is increasing on } I_k. \quad (18)$$

If  $-\infty = A$  and  $B < +\infty$  then (17) should be replaced by:

$$(B - x)Q'(x) \quad \text{is increasing on } I_k. \quad (19)$$

If both  $-\infty = A$  and  $B = +\infty$  then (17) should be ignored, thus in this case  $\exp(Q(x))$  must be convex on the whole  $I$ .

(v) for any  $x_0$  endpoint of any  $I_k$  we have:

$$\limsup_{x \rightarrow x_0^-} Q'(x) \leq \liminf_{x \rightarrow x_0^+} Q'(x), \quad (20)$$

We will simply just say that  $Q$  is weak convex on an interval  $I$  (without mentioning the basepoints), if  $Q$  is weak convex on the interval  $I = [(a, b)] \subset D$  ( $a, b \in \overline{\mathbb{R}}$ ) with basepoints  $a$  and  $b$ .

**Remark** The assumptions (16), (17), (18) and (19) are meant in the broader sense (see Notation 2 and 3).

In the next proposition we compare the "exp(Q) is convex" and the "(x - A)(B - x)Q'(x) + x is increasing" conditions. When is one weaker than the other? The answer depends on whether Q'(x) is greater or less than -1/(B - x) and 1/(x - A). The proof is not quite obvious due to the fact that differentiability of Q' is not assumed. For simplicity let A := -1 and B := 1.

**Proposition 10** *Let  $J \subset [-1, 1]$  be an open interval and  $Q$  be a function defined on  $J$  which is absolute continuous inside  $J$ .*

a) *If*

$$Q'(x) \in \left[ \frac{-1}{1-x}, \frac{1}{x+1} \right], \quad x \in J,$$

*and exp(Q(x)) is convex on J, then  $(1 - x^2)Q'(x) + x$  is increasing on J.*

b) *If*

$$Q'(x) \geq \frac{1}{x+1}, \quad x \in J, \quad (\text{or } Q'(x) \leq \frac{-1}{1-x}, \quad x \in J),$$

*and  $(1 - x^2)Q'(x) + x$  is increasing on J, then exp(Q) is convex on J.*

**Proof.**

a) Let  $q(x) := \exp(Q(x))$ ,  $x \in J$ . Let  $x, y \in J$ ,  $x < y$  be fixed. Since  $q$  is convex,  $q'(y) \geq (q(y) - q(x))/(y - x)$ , from which

$$\frac{q'(y)}{q(y)} \geq \frac{1 - \frac{q(x)}{q(y)}}{y - x}. \quad (21)$$

From the convexity it also follows that

$$q(y) \geq q(x) + (y - x)q'(x). \quad (22)$$

Notice that  $q(x) + (y - x)q'(x) > 0$ , because  $q'(x)/q(x) = Q'(x) > -1/(1 - x)$ .

Using (21) and (22) we get

$$\frac{q'(y)}{q(y)} \geq \frac{1 - \frac{q(x)}{q(x) + (y-x)q'(x)}}{y - x}. \quad (23)$$

Our goal is to show that  $(1 - x^2)Q'(x) + x \leq (1 - y^2)Q'(y) + y$ . By (23) it is enough to show that

$$(1 - x^2) \frac{q'(x)}{q(x)} + x \leq (1 - y^2) \frac{1 - \frac{q(x)}{q(x) + (y-x)q'(x)}}{y - x} + y.$$

After simplifications we see that this inequality is equivalent to

$$0 \leq \left( Q'(x) - \frac{-1}{1-x} \right) \left( \frac{1}{x+1} - Q'(x) \right),$$

which holds by the assumption of part a).

b) By symmetry we can assume that  $1/(x+1) \leq Q'(x)$ ,  $x \in J$ . The function  $k(x) := (1 - x^2)Q'(x) + x$  is increasing and  $k(x) \geq 1$  for all  $x \in J$ . Let  $x, y \in J$ ,  $x < y$  be fixed. We have

$$Q(r) = \int_x^r \frac{k(t) - t}{1 - t^2} dt + C, \quad r \in J$$

and so

$$e^{Q(r)}Q'(r) = \exp \left( \int_x^r \frac{k(t) - t}{1 - t^2} dt + C \right) \frac{k(r) - r}{1 - r^2}.$$

We want to show that  $e^{Q(x)}Q'(x) \leq e^{Q(y)}Q'(y)$ , that is,

$$1 \leq \exp \left( \int_x^y \frac{k(t) - t}{1 - t^2} dt \right) \frac{k(y) - y}{k(x) - x} \frac{1 - x^2}{1 - y^2}.$$

Since  $k$  is an increasing function, it is enough to show that

$$1 \leq \exp \left( \int_x^y \frac{K - t}{1 - t^2} dt \right) \frac{K - y}{K - x} \frac{1 - x^2}{1 - y^2}, \quad (24)$$

where the constant  $K$  is defined by  $K := k(x) \geq 1$ . The right hand side of (24) can be decreased further if we replace  $K$  by 1. But if  $K = 1$ , we have equality in (24). Statement b) is proved. ■

**Proposition 11** *Let  $J \subset \mathbb{R}$  be any open interval and  $Q$  be a function defined on  $J$  which is absolute continuous inside  $J$ . Let  $A \in \mathbb{R} \setminus J$  be arbitrary. If  $(x - A)^2Q'(x) - x$  is increasing on  $J$ , then  $\exp(Q)$  is convex on  $J$ .*

**Proof.** Assume that  $k(x) := (x - A)^2 Q'(x) - x$  is increasing on  $J$ . Let  $x, y \in J$ ,  $x < y$  be fixed. Following the idea of the proof of Proposition 10, part b), we want to show that

$$\frac{k(x) + x}{(x - A)^2} \leq \exp\left(\int_x^y \frac{k(t) + t}{(t - A)^2} dt\right) \frac{k(y) + y}{(y - A)^2}. \quad (25)$$

If  $k(y) + y \geq 0$ , then we can decrease the right hand side of (25) by replacing the function  $k(t)$  by  $K$ , and after this by replacing  $k(y)$  by  $K$ , where  $K := k(x)$ . If, however  $k(y) + y < 0$ , then we can decrease the right hand side of (25) by replacing the function  $k(t)$  by  $K^*$ , and we can increase the left hand side by replacing  $k(x)$  by  $K^*$ , where  $K^* := k(y)$ . Thus, in both cases, the new inequality we want to show is of the form

$$\frac{L + x}{(x - A)^2} \leq \exp\left(\int_x^y \frac{L + t}{(t - A)^2} dt\right) \frac{L + y}{(y - A)^2}, \quad (26)$$

where  $L$  is a constant ( $L = K$  or  $L = K^*$ ). Using the notation

$$l(r) := \int_x^r \frac{L + t}{(t - A)^2} dt$$

we see that (26) is equivalent to  $\exp(l(x))l'(x) \leq \exp(l(y))l'(y)$ , so we must show that  $\exp(l(r))$  is a convex function on  $J$ . But this is equivalent to  $(l'(r))^2 + l''(r) \geq 0$ ,  $r \in J$ , as simple differentiation shows. Now direct calculation shows that

$$(l'(r))^2 + l''(r) = \frac{(L + A)^2}{(r - A)^4} \geq 0.$$

■

**Remark** Differentiation shows that if  $Q''$  exists, then (16) holds if and only if

$$0 \leq (Q'(x))^2 + Q''(x), \quad x \in I_k,$$

and (17) holds if and only if

$$0 \leq \frac{1}{(x - A)(B - x)} + \left(\frac{1}{x - A} - \frac{1}{B - x}\right)Q'(x) + Q''(x), \quad x \in I_k, \quad (27)$$

finally, (18) holds if and only if

$$0 \leq \frac{1}{x-A}Q'(x) + Q''(x), \quad x \in I_k.$$

In most cases, the following statement is also true:

Let  $I \subset \mathbb{R}$  be an interval with  $I \subset [A_1, B_1] \subset [A_2, B_2] \subset \overline{\mathbb{R}}$ . If  $Q$  is weak convex on  $I$  with basepoints  $A_2$  and  $B_2$ , then  $Q$  is also weak convex on  $I$  with basepoints  $A_1$  and  $B_1$ . For example, this statement is true if  $Q''(x)$  exists and the sets

$$H := \left\{ x \in I : Q'(x) > \frac{1}{x-A_2} \right\} \quad \text{and} \quad H^* := \left\{ x \in I : Q'(x) > \frac{1}{B_2-x} \right\}$$

consist of finitely many intervals. To see this, let us partition  $I$  to finitely many subintervals  $J_1, J_2, \dots, J_t$  so that for any  $k$  the open subinterval  $\text{int}(J_k)$  is a subset of  $H$  or  $H^*$  completely. We will assume that both  $A_2$  and  $B_2$  are finite. (The proof is exactly the same if at least one of them is not finite. Proposition 10 can be also stated for the case when  $A$  or  $B$  is not finite, the proof remains the same.) Let us consider  $J_k$  for a fixed  $k$  ( $1 \leq k \leq t$ ). We want to show that either  $\exp(Q(x))$  is convex on  $J_k$ , or  $(x-A_1)(B_1-x)Q'(x) + x$  is increasing on  $J_k$ . Since  $k$  is arbitrary, this will prove that  $Q$  is weak convex on  $I$  with basepoints  $A_1$  and  $B_1$ . We know that either  $\exp(Q(x))$  is convex on  $J_k$ , or  $(x-A_2)(B_2-x)Q'(x) + x$  is increasing on  $J_k$ . We may assume that the second property holds, since if  $\exp(Q(x))$  is convex, then there is nothing to prove.

Case I. If  $Q'(x) \geq 1/(x-A_2)$ ,  $x \in J_k$  or  $Q'(x) \leq -1/(B_2-x)$ ,  $x \in J_k$ , then  $\exp(Q)$  must be convex on  $J_k$ . This follows from the fact that  $(x-A_2)(B_2-x)Q'(x) + x$  is increasing on  $J_k$  and from Proposition 10.

Case II. Assume now that

$$Q'(x) \in \left[ \frac{-1}{B_2-x}, \frac{1}{x-A_2} \right], \quad x \in J_k.$$

Notice that for  $x \in J_k$  we have

$$\begin{aligned} 0 \leq & \left( \frac{1}{x-A_1} - \frac{1}{x-A_2} \right) \left( \frac{1}{B_1-x} + Q'(x) \right) \\ & + \left( \frac{1}{B_1-x} - \frac{1}{B_2-x} \right) \left( \frac{1}{x-A_2} - Q'(x) \right), \end{aligned} \quad (28)$$

because all factors here are non-negative. From (28) we get

$$\begin{aligned} & \frac{1}{(x - A_2)(B_2 - x)} + \left( \frac{1}{x - A_2} - \frac{1}{B - x_2} \right) Q'(x) + Q''(x) \\ & \leq \frac{1}{(x - A_1)(B_1 - x)} + \left( \frac{1}{x - A_1} - \frac{1}{B_1 - x} \right) Q'(x) + Q''(x), \end{aligned}$$

and the left hand side of this inequality is non-negative, since  $(x - A_2)(B_2 - x)Q'(x) + x$  is increasing on  $J_k$ . Thus the right hand side is also non-negative, i.e.,  $(x - A_1)(B_1 - x)Q'(x) + x$  is increasing on  $J_k$ .

Our most general theorem which guarantees that the support is an interval is the following.

**Theorem 12** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  and suppose that  $Q$  is weak convex on the interval  $I \subset \Sigma_0$  with basepoints  $A, B \in \overline{\mathbb{R}}$  satisfying  $S_w \subset [A, B]$ . Then  $S_w \cap I$  is an interval.*

**Proof.** Suppose on the contrary that there exist  $a, b \in S_w \cap I$ ,  $a < b$ :  $(a, b) \cap S_w = \emptyset$ .

We will define a positive function  $f : (a, b) \rightarrow \mathbb{R}$  such that for any fixed  $t \in \mathbb{R} \setminus (a, b)$ :

$$g_t(x) := f(x) \left( \frac{1}{t - x} + Q'(x) \right) \quad (29)$$

is an increasing function of  $x$  on  $(a, b)$  and it is strictly increasing with the exception of finitely many  $t$ . This will finish the proof of Theorem 12 (see the beginning of the proof of Theorem 5).

Let  $I = \cup_{i=1}^n I_k$  be the decomposition of  $I$  to intervals, given in the definition of weak convexity.

Let  $E_1 := 1$ . We can find positive constants  $E_2, \dots, E_n$  uniquely such that the following  $f(x)$  function is continuous on  $I$ . For  $x \in I_k$  ( $k = 1, \dots, n$ ) let

$$f(x) := \begin{cases} E_k \exp(2Q(x)) & \text{if (16) is satisfied on } I_k \\ E_k(x - A)(B - x) & \text{if (17) is satisfied on } I_k \end{cases}. \quad (30)$$

We can assume that  $A$  and  $B$  are finite, since the proof of Theorem 12 is the the same if  $A = -\infty$  or  $B = +\infty$  with a slight modification. (For example, if  $-\infty < A$  and  $B = +\infty$ , then we have to use Theorem 8 in our proof instead of Theorem 7, and we have to define  $f(x)$  as

$$f(x) := \begin{cases} E_k \exp(2Q(x)) & \text{if (16) is satisfied on } I_k \\ E_k(x - A) & \text{if (18) is satisfied on } I_k \end{cases},$$

instead of the definition at (30).)

Now by the proof of Theorem 5, (29) is increasing on any  $I_k \cap (a, b)$  where (16) is satisfied, and strictly increasing with the exception of finitely many  $t$ . By the proof of Theorem 7, (29) is also increasing on any  $I_k \cap (a, b)$  where (17) is satisfied, and strictly increasing with the exception of finitely many  $t$ .

Let  $I_k$  and  $I_{k+1}$  to be adjacent intervals,  $I_{k+1}$  being to the right of  $I_k$ . Let  $u$  denote the number which separates  $I_k$  and  $I_{k+1}$ . Let  $x \in \text{int}(I_k) \cap (a, b)$  and  $y \in \text{int}(I_{k+1}) \cap (a, b)$  be arbitrary inner points (assuming these are not empty sets). Using what we have just said and (20) we get:

$$\begin{aligned} g_t(x) &\leq \lim_{r \rightarrow u^-} g_t(r) \leq \frac{f(u)}{t-u} + f(u) \limsup_{r \rightarrow u^-} Q'(r) \\ &= \frac{f(u)}{t-u} + f(u) \liminf_{r \rightarrow u^+} Q'(r) \leq \lim_{r \rightarrow u^+} g_t(r) \leq g_t(y). \end{aligned} \quad (31)$$

In (31) the very first and last inequality is strict inequality with the exception of finitely many  $t$ , and the set of these exceptional  $t$  values does not depend on the choice of  $x$  and  $y$ . Thus  $g_t(x) \leq g_t(y)$  and with the exception of finitely many  $t$  we have  $g_t(x) < g_t(y)$  for all  $x \in \text{int}(I_k) \cap (a, b)$  and  $y \in \text{int}(I_{k+1}) \cap (a, b)$ .

Since this holds for any  $k = 1, \dots, n$ , we conclude that  $g_t(x)$  is an increasing function of  $x$  on the whole  $(a, b)$  and it is strictly increasing with the exception of finitely many  $t$ . ■

**Examples** Since our weak convexity definition is a generalization of the one given in [1], all the examples given in [1] could be repeated here. We just list some new examples to demonstrate how large the class of weak convex functions is.

1.) Let  $C(x)$  be any positive convex function on  $\Sigma := \mathbb{R}$ . Then  $Q(x) := \ln(C(x))$  is weak convex on  $\Sigma := \mathbb{R}$ . If we assume that  $\lim_{|x| \rightarrow \infty} x/C(x) = 0$ , then  $w(x) := 1/C(x)$  is an admissible weight function, and  $S_w$  will be an interval by Theorem 12.

2.)  $Q(x) := \frac{1+c}{2} \ln(1+x) + \frac{1-c}{2} \ln(1-x)$  is weak convex on  $\Sigma := [-H, H]$ , where  $c$  is any constant and  $H \in (0, 1)$ . Thus  $S_w$  is an interval. Now

$$w(x) = \frac{1}{(1-x)^{(1-c)/2}(1+x)^{(1+c)/2}}, \quad x \in [-H, H]$$

is the corresponding weight function. In particular if  $c := 0$  we get that the concave function  $Q(x) := \ln \sqrt{1-x^2}$  is weak convex on  $[-H, H]$  (now  $w(x) = 1/\sqrt{1-x^2}$ ). (We remark that  $\exp(Q(x))$  is also concave.)

Notice that  $(1-x^2)Q'(x) + x = c$  constant, which shows that our  $Q(x)$  in example 2.) is weak convex with basepoints  $-1$  and  $1$ . Hence by the remark on Page 14 it is weak convex indeed.

3.) For any  $c > 0$ ,  $Q(x) := \cos(cx)/c^2$  is weak convex on  $\Sigma := [-1, 1]$ , thus  $S_w$  is an interval.

Indeed,  $(1-x^2)Q'(x) + x$  is increasing, because  $Q'(x) = -\sin(cx)/c$ , and

$$\frac{d}{dx}[(1-x^2)Q'(x) + x] = \frac{2x \sin(cx)}{c} - (1-x^2) \cos(cx) + 1 \geq \frac{2x \sin(cx)}{c} + x^2 \geq 0.$$

4.)  $Q(x) := x^2(x-1/2)(x+1/3)$  is weak convex on  $\Sigma := [-1, 1]$ , thus  $S_w$  is an interval.

In this example neither  $\exp(Q)$  is convex on the whole  $[-1, 1]$ , nor  $(1-x^2)Q'(x) + x$  is increasing on the whole  $[-1, 1]$ . However,  $\exp(Q)$  is convex on  $[-1, -0.2] \cup [0.3, 1]$  and  $(1-x^2)Q'(x) + x$  is increasing on  $[-0.7, 0.7]$ . So we can partition  $[-1, 1]$  to smaller intervals (say, to  $[-1, -0.2]$ ,  $[-0.2, 0.7]$  and  $[0.7, 1]$ ) so that one of our criteria is satisfied on any of the small intervals, as it is described at the definition of weak convexity. Hence  $Q(x)$  is weak convex.

5.) In [2], [3] and [7] Damelin, Dragnev and Kuijlaars studied the following external fields (sometimes in the context of fast decreasing polynomials):

$$Q(x) := -cx^\alpha, \quad x \in \Sigma := [0, 1], \quad (32)$$

and

$$Q(x) := -c \operatorname{sign}(x)|x|^\alpha, \quad x \in \Sigma := [-1, 1], \quad (33)$$

where  $c > 0$ ,  $\alpha \geq 1$ . They proved that for the external fields at (32) and at (33) the support will be the union of at most two intervals. In both cases  $S_w$  is the full interval ( $S_w = [0, 1]$  and  $S_w = [-1, 1]$  respectively), whenever  $c$  is small enough.

In this direction we can say the following: Let  $q(x)$  be a twice differentiable function on  $\Sigma := [-1, 1]$  such that  $q''(x)$  is bounded and let  $Q(x) := cq(x)$ ,  $x \in [-1, 1]$  where  $c \in \mathbb{R}$ . Then there exists  $\epsilon > 0$  such that  $S_w$  is an interval whenever  $|c| < \epsilon$ . The proof of this statement is simple: if  $c$  is close to zero, then (27) holds, say, with  $A := -2, B := 2$ , so

from Theorem 7 we gain that  $S_w$  is an interval (but not necessarily the full  $[-1, 1]$ ).

Let us examine now the case when  $Q(x) := -cx^\alpha$ ,  $x \in \Sigma := [0, 1]$ . Let  $A := 0, B := 1$ , so  $g(x) := (x - A)(B - x)Q'(x) + x = -c\alpha(x - x^2)x^{\alpha-1} + x$ . Then  $g'(x) := -c\alpha^2x^{\alpha-1} + c\alpha(\alpha + 1)x^\alpha + 1$ . Notice that  $g'(0) = 1$  and  $g'(1) = c\alpha + 1$ , which are positive values. Calculating  $g''(x)$  we see that  $g'(x)$  is a decreasing function on  $[0, (\alpha - 1)/(\alpha + 1)]$  and increasing on  $[(\alpha - 1)/(\alpha + 1), 1]$ . So there exist  $G_1, G_2 \in [0, 1]$ ,  $G_1 \leq G_2$ , such that  $g'(x)$  is non-negative on  $[0, G_1] \cup [G_2, 1]$  and negative on  $(G_1, G_2)$ . That is,  $g(x)$  is increasing on  $[0, G_1] \cup [G_2, 1]$ , so both  $S_w \cap [0, G_1]$  and  $S_w \cap [G_2, 1]$  are intervals (empty sets are possible). But our criteria (17) cannot state anything on  $[G_1, G_2]$ . Using (16), too, we see that  $\exp(Q)$  is convex on  $[G, 1]$  where  $G := (\alpha - 1)/(c\alpha)$ . Thus  $S_w \cap [0, G_1]$  and  $S_w \cap [\min(G_2, G), 1]$  are intervals (empty sets are possible), but unfortunately we cannot say anything about  $S_w \cap [G_1, \min(G_2, G)]$ .

For the sake of simple notations we state the next theorem for the interval  $[-1, 1]$ , instead of  $[A, B]$ . (To get the general version, replace everywhere  $x+1$  by  $x - A$  and  $1 - x$  by  $B - x$ .)

**Theorem 13** *Let  $w = \exp(-Q)$  be an admissible weight on  $\mathbb{R}$  such that  $\min S_w = -1$ ,  $\max S_w = 1$ . Suppose that  $Q$  is absolutely continuous inside  $(-1, 1)$  and  $Q'$  is bounded on  $[-1, 1]$ . Assume further that  $\liminf_{x \rightarrow -1+0} Q(x) = Q(-1)$ ,  $\liminf_{x \rightarrow 1-0} Q(x) = Q(1)$  and with some constants  $-1 \leq u \leq v \leq 1$  we have*

$$\begin{aligned} Q'(x) &\leq \frac{-1}{1-x} && \text{on } (-1, u) \\ \frac{-1}{1-x} &\leq Q'(x) \leq \frac{1}{x+1} && \text{on } (u, v) \\ \frac{1}{x+1} &\leq Q'(x) && \text{on } (v, 1), \\ (1-x)^2 Q'(x) - x &&& \text{is increasing on } (-1, u). \\ (1-x^2) Q'(x) + x &&& \text{is increasing on } (u, v). \\ (x+1)^2 Q'(x) - x &&& \text{is increasing on } (v, 1). \end{aligned}$$

Then  $S_w = [-1, 1]$  and the density of  $\mu_w$  is  $d\mu_w(t) = v(t)dt$  a.e.  $t \in [-1, 1]$ , where we define  $v(t)$  for a.e.  $t \in [-1, 1]$  as follows:

For  $t \in (-1, u)$ , let

$$v(t) := \frac{1}{\pi^2 \sqrt{1-t^2}} \left[ \frac{t+1}{1-t} \int_{-1}^1 \frac{(1-s)^2 Q'(s) - s - [(1-t)^2 Q'(t) - t]}{(s-t)\sqrt{1-s^2}} ds \right. \\ \left. + \frac{2}{1-t} \left( \pi + \int_{-1}^1 \frac{(1-s)Q'(s)}{\sqrt{1-s^2}} ds \right) \right] \quad (34)$$

For  $t \in (u, v)$ , let

$$v(t) := \frac{1}{\pi^2 \sqrt{1-t^2}} \int_{-1}^1 \frac{(1-s^2)Q'(s) + s - [(1-t^2)Q'(t) + t]}{(s-t)\sqrt{1-s^2}} ds. \quad (35)$$

For  $t \in (v, 1)$ , let

$$v(t) := \frac{1}{\pi^2 \sqrt{1-t^2}} \left[ \frac{1-t}{t+1} \int_{-1}^1 \frac{(s+1)^2 Q'(s) - s - [(t+1)^2 Q'(t) - t]}{(s-t)\sqrt{1-s^2}} ds \right. \\ \left. + \frac{2}{t+1} \left( \pi - \int_{-1}^1 \frac{(s+1)Q'(s)}{\sqrt{1-s^2}} ds \right) \right] \quad (36)$$

**Proof.** The hypotheses of Theorem 13 imply that  $\exp(Q)$  is convex on  $(-1, u)$  and on  $(v, 1)$  (see Proposition 11). Thus  $Q$  is weak convex on  $[-1, 1]$  and so  $S_w$  is an interval, which must be  $[-1, 1]$ .

According to [1] (Lemma 16), if the function

$$h(t) := \frac{1}{\pi^2 \sqrt{1-t^2}} PV \int_{-1}^1 \frac{\sqrt{1-s^2} Q'(s)}{s-t} ds + \frac{1}{\pi \sqrt{1-t^2}} \quad (37)$$

is non-negative (a.e.  $t \in [-1, 1]$ ), then  $h(t)$  is the density of  $\mu_w$ .

We perform the following manipulation:

$$(t+1)[(1-s)^2 Q'(s) - s] + 2[(1-s)(s-t)Q'(s) + s] = (1-t)[(1-s^2)Q'(s) + s] \quad (38)$$

Dividing this by  $(s-t)\sqrt{1-s^2}$  and integrating with respect to  $s$  (principal value integral), we get:

$$\frac{1}{\pi^2 \sqrt{1-t^2}} \left[ \frac{t+1}{1-t} PV \int_{-1}^1 \frac{(1-s)^2 Q'(s) - s}{(s-t)\sqrt{1-s^2}} ds \right.$$

$$+\frac{2}{1-t}\left(\pi + \int_{-1}^1 \frac{(1-s)Q'(s)}{\sqrt{1-s^2}} ds\right) = h(t) \quad (39)$$

where we used the fact (see [9], formula IV.(3.20)) that

$$PV \int_{-1}^1 \frac{1}{(s-t)\sqrt{1-s^2}} ds = 0 \quad (40)$$

and

$$\begin{aligned} & PV \int_{-1}^1 \frac{s}{(s-t)\sqrt{1-s^2}} ds \\ &= \int_{-1}^1 \frac{1}{\sqrt{1-s^2}} ds + t PV \int_{-1}^1 \frac{1}{(s-t)\sqrt{1-s^2}} ds = \pi + 0 = \pi. \end{aligned} \quad (41)$$

From (37) and (41) we also see that

$$\frac{1}{\pi^2 \sqrt{1-t^2}} PV \int_{-1}^1 \frac{(1-s^2)Q'(s) + s}{(s-t)\sqrt{1-s^2}} ds = h(t). \quad (42)$$

Finally, using the identity

$$(1-t)[(s+1)^2 Q'(s) - s] - 2[(s+1)(s-t)Q'(s) - s] = (t+1)[(1-s^2)Q'(s) + s]$$

we get the same way that

$$\begin{aligned} & \frac{1}{\pi^2 \sqrt{1-t^2}} \left[ \frac{1-t}{t+1} PV \int_{-1}^1 \frac{(s+1)^2 Q'(s) - s}{(s-t)\sqrt{1-s^2}} ds \right. \\ & \left. + \frac{2}{t+1} \left( \pi - \int_{-1}^1 \frac{(s+1)Q'(s)}{\sqrt{1-s^2}} ds \right) \right] = h(t) \end{aligned} \quad (43)$$

Now we claim that  $h(t)$  is non-negative for a.e.  $t \in [-1, 1]$ . Assuming  $-1 \neq u$ , let  $t \in (-1, u)$ . Notice that in (39) the term

$$\pi + \int_{-1}^1 \frac{(1-s)Q'(s)}{\sqrt{1-s^2}} ds$$

is non-negative, see [1], Theorem 10.

Actually, Theorem 10 in [1] has different assumptions on  $Q$  than our assumptions at Theorem 13 above. Nevertheless, the proof of Theorem 10

([1]) works for our assumptions, if we make the following modification in the proof. Let  $A := -1, B := 1, a \in [A, B)$  and let

$$h(\beta) := \int_a^\beta \sqrt{\frac{x-a}{\beta-x}} Q'(x) dx, \quad \beta \in (a, B].$$

We have to show that  $h(\beta)$  is continuous on  $(a, B]$  - which is a long argument in the proof of Theorem 10 ([1]). But now, using the boundedness of  $Q'$ , we can see this immediately. For this purpose let  $\beta_0, \beta \in (a, B]$  and let  $\beta \rightarrow \beta_0$ . Without loss of generality we can assume that  $\beta_0 < \beta$ . Using the triangle inequality we get

$$\begin{aligned} |h(\beta) - h(\beta_0)| &\leq \left| \int_{\beta_0}^\beta \sqrt{\frac{x-a}{\beta-x}} Q'(x) dx \right| + \left| \int_a^{\beta_0} \left[ \sqrt{\frac{x-a}{\beta-x}} - \sqrt{\frac{x-a}{\beta_0-x}} \right] Q'(x) dx \right| \\ &\leq C \left[ \int_{\beta_0}^\beta \sqrt{\frac{1}{\beta-x}} dx + \int_a^{\beta_0} \left[ \sqrt{\frac{1}{\beta_0-x}} - \sqrt{\frac{1}{\beta-x}} \right] dx \right], \end{aligned}$$

and clearly both of these terms are going to zero as  $\beta \rightarrow \beta_0$ .

Let us return to the proof of Theorem 13. Let  $t$  be a number such that  $Q'(t)$  exists. In (39) the term

$$\begin{aligned} &PV \int_{-1}^1 \frac{(1-s)^2 Q'(s) - s}{(s-t)\sqrt{1-s^2}} ds \\ &= PV \int_{-1}^1 \frac{(1-s)^2 Q'(s) - s - [(1-t)^2 Q'(t) - t]}{(s-t)\sqrt{1-s^2}} ds \end{aligned} \quad (44)$$

is also non-negative which we can see as follows: if  $s \in (-1, u)$ , then

$$\frac{(1-s)^2 Q'(s) - s - [(1-t)^2 Q'(t) - t]}{s-t}$$

is clearly non-negative because  $(1-x)^2 Q'(x) - x$  is increasing on  $(-1, u)$ . Now let  $s \in (u, 1)$  (if  $u \neq 1$ ). By the assumptions of Theorem 13 we have  $Q'(t) \leq (-1)/(1-t)$  and  $Q'(s) \geq (-1)/(1-s)$ , from which

$$\begin{aligned} &\frac{(1-s)^2 Q'(s) - s - [(1-t)^2 Q'(t) - t]}{s-t} \\ &\geq \frac{(1-s)^2 \frac{-1}{1-s} - s - [(1-t)^2 \frac{-1}{1-t} - t]}{s-t} = 0 \end{aligned}$$

Since the integrand at the right hand side of (44) is non-negative it is no longer a principal value integral but a Lebesgue integral. So the "PV" sign can be dropped.

The proof of the non-negativity of  $h(t)$  in the remaining cases (when  $t \in (u, v)$  or  $t \in (v, 1)$ ) is similar. Thus  $h(t)$  is the density, indeed. The integral formulas in Theorem 13 are coming from (39), (42) and (43) immediately (by using (40)).



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