

BIPARTITE PARTIAL DUALS AND CIRCUITS IN MEDIAL GRAPHS

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ABSTRACT. It is well known that a plane graph is Eulerian if and only if its geometric dual is bipartite. We extend this result to partial duals of plane graphs. We then characterize all bipartite partial duals of a plane graph in terms of oriented circuits in its medial graph.

1. INTRODUCTION AND STATEMENTS OF RESULTS

The geometric dual, G^* , of an embedded graph G is a fundamental construction in graph theory and appears in many places throughout mathematics. Motivated by various new constructions in knot theory, S. Chmutov, in [2], introduced the concept of the partial dual of an embedded graph. Roughly speaking, the partial dual of an embedded graph is obtained by forming the geometric dual with respect to only a subset of edges of the graph (a formal definition is given Subsection 2.3). Partial duality appears to be a fundamental operation on embedded graphs and, although it has only recently been introduced, it has found a number of applications in graph theory, topology, and physics (see, for example, [2, 4, 5, 7, 8, 9, 10, 11, 12, 13]). While geometric duality always preserves the surface in which a graph is embedded, this is not the case for the more general partial duality. For example, if G is a plane graph, then G^* is also a plane graph, but a partial dual G^A of G need not be plane. Partially dual embedded graphs can have different topological and graph theoretical properties.

Rather than being concerned with the ways in which a graph and its partial dual can differ, here we are interested in how partial duality both preserves and transforms the structure of an embedded graph. In particular, we determine the extent to which partial duality preserves the following classical connection between Eulerian and bipartite plane graphs.

Theorem 1. *Let G be a plane graph, then G is Eulerian if and only if its dual, G^* , is bipartite.*

This theorem is known to hold more generally for binary matroids (see [14], and also [6]), but it does not hold for non-plane graphs (although the geometric dual of a bipartite graph is always Eulerian). Here we give the extension of this classical connection between Eulerian and bipartite graphs from geometric duality to partial duality. We prove:

Theorem 2. *Let G be a plane graph and $A \subseteq E(G)$. Then:*

- (1) G^A is bipartite if and only if the components of $G|_A$ and $G^*|_{A^c}$ are Eulerian;
- (2) G^A is Eulerian if and only if $G|_A$ and $G^*|_{A^c}$ are bipartite.

This result appears in Section 3 as Theorem 6. Its proof requires much more work than the case for geometric duality stated in Theorem 1. To prove the result we introduce a new way of obtaining

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the underlying abstract graph of partial dual G^A (see Theorem 5 below). The advantage of this construction is that it avoids having to construct the embedded graph itself.

Having established the relation between Eulerian and bipartite partial duals, we then turn our attention to the problem of determining which subsets of edges in a graph give rise to bipartite and Eulerian partial duals. That is, given a plane graph G , we characterize the subsets $A \subseteq E(G)$ having the property that G^A is bipartite or Eulerian. It turns out that this problem is intimately related to oriented circuits in the medial graph G_m of G . We provide the following complete characterization of edge sets that lead to bipartite partial duals:

Theorem 3. *Let G be a plane graph. Then the partial dual G^A is bipartite if and only if A is the set of c -edges arising from an all-crossing direction of G_m .*

The terminology for this theorem, together with its proof, appears in Section 4. (Figure 2 offers a quick indication of the terminology). We note that Theorem 2 is used in an essential way to prove this characterization. We also find a sufficient condition for set of edges to give rise to an Eulerian partial dual in terms of circuits in medial graphs (see Corollary 2). Some connections of our results with knot theory are discussed in Remark 4.

2. EMBEDDED GRAPHS AND DUALITY

2.1. Cellularly embedded graphs and ribbon graphs. We begin with a brief review of embedded graphs and ribbon graphs. Note that we will be concerned with both cellularly and non-cellularly embedded graphs.

An *embedded graph* $G = (V(G), E(G)) \subset \Sigma$ is a graph drawn on a surface Σ in such a way that edges only intersect at their ends. The arcwise-connected components of $\Sigma \setminus G$ are called the *regions* of G . If each of the regions of an embedded graph G is homeomorphic to a disc we say that G is a *cellularly embedded graph*, and its regions are called *faces*. A *plane graph* is a graph that is cellularly embedded in the sphere (rather than the plane).

Two embedded graphs, $G \subset \Sigma$ and $G' \subset \Sigma'$, are said to be *equal* if there is a homeomorphism from Σ to Σ' that sends G to G' . As is common, we will often abuse notation and identify an embedded graph with its equivalence class under equality.

We will need to work with cellularly embedded graphs which arise as subgraphs of cellularly embedded graphs. Accordingly, we will often find it particularly convenient and natural to describe embedded graphs as ribbon graphs.

Definition 1. A *ribbon graph* $G = (V(G), E(G))$ is a (possibly non-orientable) surface with boundary represented as the union of two sets of topological discs: a set $V(G)$ of *vertices*, and a set of *edges* $E(G)$ such that

- (1) the vertices and edges intersect in disjoint line segments;
- (2) each such line segment lies on the boundary of precisely one vertex and precisely one edge;
- (3) every edge contains exactly two such line segments.

It is well known and easily seen that ribbon graphs are equivalent to cellularly embedded graphs. Intuitively, if G is a cellularly embedded graph, a ribbon graph representation results from taking a small neighbourhood of G . Neighbourhoods of vertices of G form the vertices of the ribbon graph, and neighbourhoods of the edges of G form the edges of the ribbon graph. On the other hand, if G is a ribbon graph, we simply sew discs into each boundary component of the ribbon graph to get a graph cellularly embedded in a surface. Since ribbon graphs and cellularly embedded graphs are equivalent we can, and will, move freely between them.

Two ribbon graphs are considered to be equal if their corresponding embedded graphs are equal. This means that two ribbon graphs are equal if there is a homeomorphism between their underlying

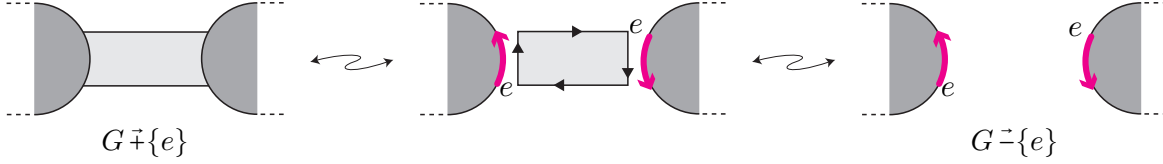


FIGURE 1. Constructing $G^-{e}$ and $G^+{e}$.

surfaces that preserve the vertex-edge structure. Again, at times we abuse notation and identify a ribbon graph with its equivalence class under equality.

Just as with graphs, if G is a ribbon graph and $A \subseteq E(G)$, then $G - A$ is the ribbon graph obtained from G by deleting all of the edges in A . Note that $G - A$ is also a ribbon graph and therefore describes a cellularly embedded graph. It is this closure of the set of ribbon graphs under deletion of edges that makes them useful here; note that deleting edges in a cellularly embedded graph may result in a non-cellularly embedded graph. Furthermore, if G is a ribbon graph and $A \subseteq E(G)$, then we let $G|_A$ denote the ribbon subgraph of G whose edge set is A and whose vertex set consists of all vertices of G which are incident to an edge in A . If G is a cellularly embedded graph then $G|_A$ is defined to be the cellularly embedded graph corresponding to this ribbon graph.

We will need to be able to delete edges from a ribbon graph without losing any information about the position of the edge. We will do this by recording the position of the edge using labelled arrows.

Definition 2. An *arrow-marked ribbon graph* consists of a ribbon graph equipped with a collection of coloured arrows, called *marking arrows*, on the boundaries of its vertices. The marking arrows are such that no marking arrow meets an edge of the ribbon graph, and there are exactly two marking arrows of each colour.

Let G be a ribbon graph and $A \subseteq E(G)$. Then we let G^-A denote the arrow-marked ribbon graph obtained, for each edge $e \in A$, as follows: arbitrarily orient the boundary of e ; place an arrow on each of the two arcs where e meets vertices of G , such that the directions of these arrows follow the orientation of the boundary of e ; colour the two arrows with e ; and delete the edge e . This process is illustrated, locally at an edge, in Figure 1.

Conversely, given an arrow-marked ribbon graph G with set of labels A , we can recover a ribbon graph G^+A as follows: for each label $e \in A$ take a disc and orient its boundary arbitrarily; add this disc to the ribbon graph by choosing two non-intersecting arcs on the boundary of the disc and two marking arrows of the same colour, and then identifying the arcs with the marking arrows according to the orientation of the arrow. The disc that has been added forms an edge of a new ribbon graph. Again, this process is illustrated in Figure 1.

Example 1. This example illustrates how a ribbon graph gives rise to an arrow-marked ribbon graph.



Note that G can be recovered from the arrow-marked ribbon graph G^-A , shown on the right, by taking $A = \{1, 2, 5\}$ to be the set of labels and forming $(G^-A)^+A$.

From the above we see that every arrow-marked ribbon graph gives rise to a ribbon graph. We then say that two arrow-marked ribbon graphs are considered to be equal if the ribbon graphs they describe are equal. We will generally abuse notation and regard the set of labels of an arrow-marked ribbon graph as a set of edges. This will allow us to view A as an edge set in expressions like $G = (G \dashv A) \mp A$.

2.2. Geometric duals. The construction of the *geometric dual*, G^* , of a cellularly embedded graph $G \subset \Sigma$ is well known: G^* is formed by placing one vertex in each face of G and embedding an edge of G^* between two vertices whenever the faces of G they lie in are adjacent. Observe that G^* has a natural cellular embedding in Σ , and that there is a natural (cellular) immersion of $G \cup G^*$ where each edge of G intersects exactly one edge of G^* at exactly one point. We will call this immersion the *standard immersion* of $G \cup G^*$.

There is a natural bijection between $E(G)$ and $E(G^*)$. We will generally use this bijection to identify the edges of G and the edges of G^* . However, at times we will be working with $G \cup G^*$, so to avoid confusion we will use e^* to denote the edge of G^* which corresponds to the edge e of G .

Geometric duals have a particularly neat description in the language of ribbon graphs. Let $G = (V(G), E(G))$ be a ribbon graph. We can regard G as a punctured surface. By filling in the punctures using a set of discs denoted $V(G^*)$, we obtain a surface without boundary, Σ . The *geometric dual* of G is the ribbon graph $G^* = (V(G^*), E(G))$.

Suppose now that G is an arrow-marked ribbon graph, so that G has labelled arrows on its vertices. Then in the formation of G^* as described above, the boundaries of the vertices of G and G^* intersect, and therefore the marking arrows on G induce marking arrows on G^* . The geometric dual, G^* , of an arrow-marked ribbon graph, G , is the geometric dual of the underlying ribbon graph equipped with the induced marking arrows.

Note that for ribbon graphs geometric duality acts disjointly on connected components, so that $(G \sqcup H)^* = G^* \sqcup H^*$.

We will also need to form geometric duals of non-cellularly embedded graphs. Since the properties of duality depend upon whether or not a graph is cellularly embedded, we will avoid confusion by denoting the dual of a not necessarily cellularly embedded graph by G^\circledast . The embedded graph G^\circledast is formed just as the geometric dual of an embedded graph is formed but by placing a vertex in each region of G , rather than each face. That is, if $G \subset \Sigma$ is an embedded graph (the embedding may or may not be cellular here), then $G^\circledast \subset \Sigma$ is the embedded graph formed by placing one vertex in each region of G , and embedding an edge of G^\circledast between two vertices whenever the regions of G they lie in are adjacent. It is important to note that in general $(G^\circledast)^\circledast \neq G$.

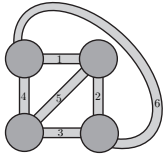
2.3. Partial duals of embedded graphs. We can now describe partial duality, which was introduced by S. Chmutov in [2] to unify various results which realize the Jones polynomial as a graph polynomial (two of these results were first related in [8]). We will use the definition of a partial dual from [10]. Chmutov's original (and equivalent) definition of a partial dual can be found in [2].

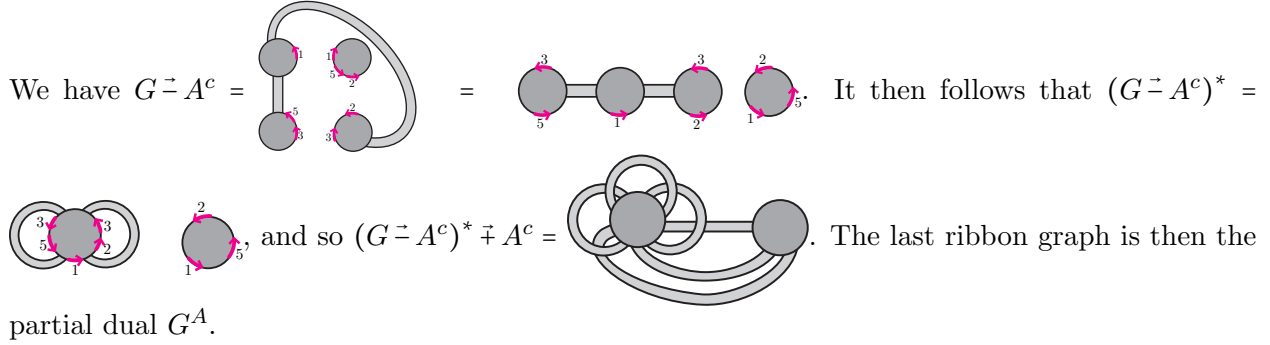
Let G be a ribbon graph and $A \subseteq E(G)$. Then the partial dual G^A of G is formed by: 'hiding' the edges that are not in A by replacing them with marking arrows using $G \dashv A^c$; forming the geometric dual $(G \dashv A^c)^*$ (so that the dual is only taken with respect to the edges of G that are in A); then putting back in the edges that are not in A , giving $(G \dashv A^c)^* \mp A$. The resulting ribbon graph is G^A . Here and henceforth, A^c denotes the complementary edge set $E(G) - A$ of A . This process is summarized by the following definition.

Definition 3. Let G be a ribbon graph and $A \subseteq E(G)$. Then the *partial dual of G with respect to A* , denoted by G^A , is given by

$$G^A := (G \dashv A^c)^* \mp A^c.$$

The partial dual of a cellularly embedded graph is obtained by translating into the language of ribbon graphs, forming the partial dual, and translating back into the language of cellularly embedded graphs.

Example 2. In this example we will form the partial dual G^A where $G =$ , and $A = \{4, 6\}$.



Further examples of partial duals can be found in [2, 9, 10, 11], and the references therein.

We will need the following basic properties of partial duality later. These properties are due to Chmutov and can be found in [2].

Proposition 1. *Let G be a ribbon graph and $A, B \subseteq E(G)$. Then*

- (1) $G^\emptyset = G$;
- (2) $G^{E(G)} = G^*$, where G^* is the geometric dual of G ;
- (3) $(G^A)^B = G^{A\Delta B}$, where $A\Delta B := (A \cup B) \setminus (A \cap B)$ is the symmetric difference of A and B .

3. EULERIAN AND BIPARTITE PARTIAL DUALS

This section gives our first main result (the second being Theorem 7), which is the extension to partial duality of the following classical result.

Theorem 4. *Let G be a graph embedded in the plane, then G is Eulerian if and only if G^\otimes is bipartite.*



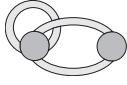



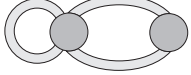
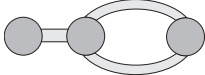


This theorem usually includes the restriction that G is connected, and therefore a plane graph, but we need this slightly more general form.

We begin with the observation that biparticity is a property of abstract graphs rather than embedded graphs. Accordingly, in order to study the 2-colourability of partially dual embedded graphs, it suffices to study their underlying abstract graphs. This allows us to use tools developed in [10] for partial duals of abstract graphs to prove Theorem 6. Also, to prove the theorem, we introduce a new way of constructing abstract graphs that are partial duals.

Recall that an embedded graph G consists of an embedding of a graph \hat{G} into a surface. We call the graph \hat{G} the *underlying abstract graph* of G . If G and H are two embedded graphs with the same underlying abstract graphs we will say that G and H are *equivalent as abstract graphs* and write $G \cong H$. The notion of partially dual abstract graphs was introduced in [10].

Definition 4. Two abstract graphs are said to be *partial duals* if they are the underlying abstract graphs of two partially dual embedded graphs.

Remark 1. It is important to observe that although partial duality is a transitive relation for embedded graphs, it is not a transitive relation for abstract graphs. For example, the two abstract

graphs  and  are partial duals, because they are the underlying abstract graphs of the partial duals  and  respectively. Also, the abstract graphs  and  are partial duals, since they are the underlying abstract graphs of  and  respectively. However,  and  are not partially dual abstract graphs. This observation has implications for the results presented here.

We now give a new way of constructing partially dual abstract graphs. Theorem 6 will follow easily from this construction. Given an embedded graph G , Theorem 5 provides a way to obtain an embedded graph that is equivalent to G^A as an abstract graph but not necessarily as an embedded graph. This result is especially useful here since G^A is bipartite if and only if any embedded graph H with $H \cong G^A$ is bipartite, so we need not worry about the embedding of G^A .

Since we will be working simultaneously with an embedded graph G and its dual G^* , we will use a superscript ‘*’ to denote corresponding edges and edge sets in G^* . For example, if e is an edge in G then e^* denotes the edge in G^* under the natural identification of $E(G)$ with $E(G^*)$.

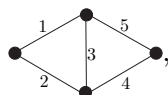
Theorem 5. *Let G be a connected, cellularly embedded graph, and $A \subseteq E(G)$. Then*

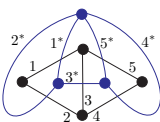
$$[(G \cup G^*) - (A^c \cup A^*)]^\otimes \cong G^A.$$

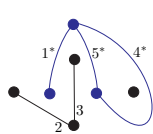
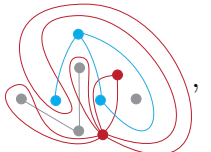
Here, $G \cup G^*$ has the standard immersion.

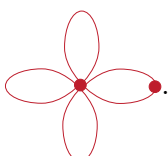
In order not to disrupt our narrative on bipartite partial duals, we defer the somewhat technical proof of Theorem 5 until Section 5.

We emphasize the fact that in general $[(G \cup G^*) - (A^c \cup A^*)]^\otimes$ and G^A are not equal as embedded graphs. This is what makes the above theorem significant: we have found a way of constructing partial duals of abstract graphs that does not require us to pass through partially dual ribbon graphs. It is perhaps prudent to highlight a second point, that in general $[(G \cup G^*) - (A^c \cup A^*)]^\otimes$ and $(G^A)^*$ are not isomorphic as abstract graphs. This means that, for a plane graph G , if $[(G \cup G^*) - (A^c \cup A^*)]^\otimes \cong G^A$ is bipartite (respectively Eulerian), then although the dual $[(G \cup G^*) - (A^c \cup A^*)]^\otimes$ is Eulerian (respectively bipartite), we do not know whether $(G^A)^*$ is bipartite or Eulerian.

Example 3. The following example illustrates Theorem 5. Let $G =$ , and $A = \{2, 3\}$.

Then $G \cup G^* =$ , and we have $A^c = \{1, 4, 5\}$ and $A^* = \{2^*, 3^*\}$. Thus $(G \cup G^*) - (A^c \cup A^*) =$

. The dual of this non-cellularly embedded graph is formed as follows ,

giving $[(G \cup G^*) - (A^c \cup A^*)]^\otimes =$ .

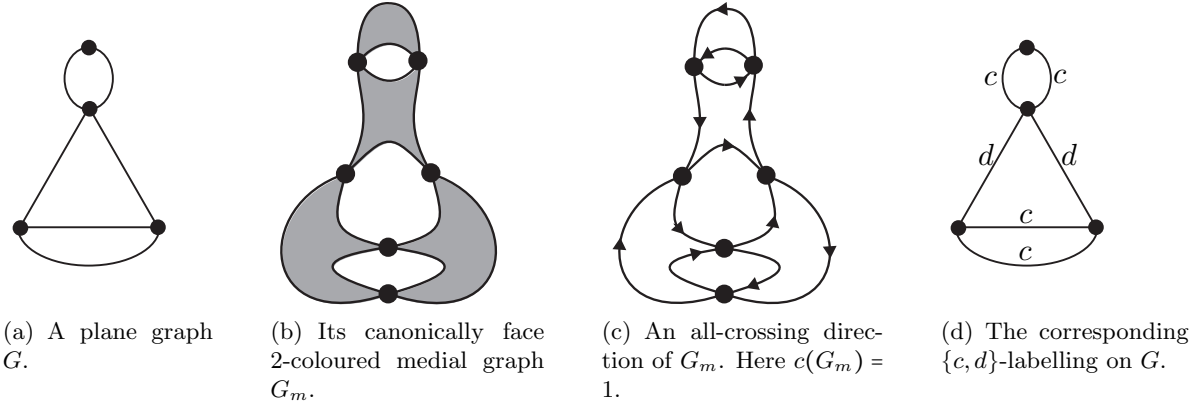
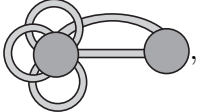
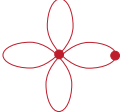


FIGURE 2. Examples of constructions associated with medial graphs.

On the other hand, by regarding G as a genus zero ribbon graph, we can form the partial dual G^A to obtain $G^A =$ , which is isomorphic to  as an abstract graph.

Our generalization of Theorem 4 is the following theorem, in which we identify the edges of G and G^A in the standard way.

Theorem 6. *Let G be a plane graph and $A \subseteq E(G)$. Then:*

- (1) G^A is bipartite if and only if the components of $G|_A$ and $G^*|_{A^c}$ are Eulerian;
- (2) G^A is Eulerian if and only if $G|_A$ and $G^*|_{A^c}$ are bipartite.

Note that Theorem 4 is obtained from Theorem 6 by setting $A = \emptyset$. Also note that, as one would expect, Theorem 6 does not hold for graphs embedded in higher genus surfaces.

Proof of Theorem 6. Let G be a plane graph, and set $\Phi := (G \cup G^*) - (A^c \cup A^*)$. By Theorem 5, G^A is bipartite if and only if each component of Φ^* is bipartite. But, as Φ is embedded in the plane, Theorem 4 implies that each component of Φ^* is bipartite if and only if each component of Φ is Eulerian. This happens if and only if each component of $G - A^c$ and of $G^* - A^*$ is Eulerian, which happens if and only if each component of $G|_A$ and of $G^*|_{A^c}$ is Eulerian. This argument can be restated with the words bipartite and Eulerian exchanged, thus completing the proof. \square

Remark 2. The obvious extension to Theorem 4 is that, for a plane graph G , G^A is bipartite if and only if $(G^A)^*$ is Eulerian. However, this statement is not true in one direction. For example, if G^A is the cellularly embedded graph on a torus that consists of a loop added to a 2-cycle, then G^A is Eulerian, but $(G^A)^*$ is not bipartite. The statement is true in the other direction since the dual of any bipartite embedded graph is Eulerian (see Remark 3).

4. BIPARTITE GRAPHS AND CIRCUITS IN MEDIAL GRAPHS

In this section we apply Theorem 6 to obtain our second main result, which is a characterization of those edge sets of plane graphs which give rise to bipartite partial duals. This characterization will be in terms of circuits in the medial graph.

4.1. Medial graphs. The *medial graph* G_m of a plane graph G is the 4-regular plane graph obtained from G by placing a vertex on each edge of G , and joining two such vertices by an edge embedded in a face whenever the two edges on which they lie are on adjacent edges of the face.



FIGURE 3. The definition of c -vertices and d -vertices.

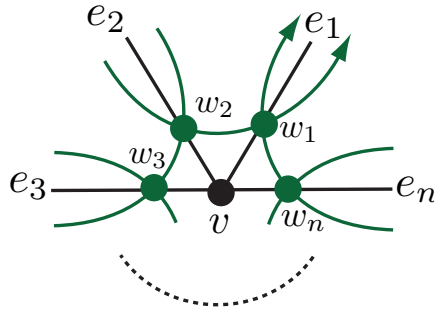


FIGURE 4. Walking around vertex v of graph G .

Each vertex of G corresponds to a face of G_m . If we colour all such faces of G_m black and the remaining faces white we obtain a face 2-colouring of G_m which we will call the *canonical face 2-colouring*. (See Figures 2(a) and 2(b).)

We are interested in particular directed graphs which arise by directing the edges of a medial graph. An *all-crossing direction* of G_m is an assignment of a direction to each edge of G_m in such a way that at each vertex v of G_m , when we follow the cyclic order of the directed edges incident to v , we find (head, head, tail, tail). (See Figure 2(c).)

We will let $c(G_m)$ denote the number of circuits in any all-crossing direction of G_m which are obtained by following the directed edges in such a way that at each vertex, we enter and exit at a head and tail which are not adjacent in the cyclic order of the incident edges at that vertex. (See Figure 2(c).) We observe that $c(G_m)$ is independent of the choice of all-crossing direction of G_m , and that G_m admits $2^{c(G_m)}$ all-crossing directions.

If G_m is equipped with the canonical face 2-colouring then we can partition the vertices of G_m by calling each vertex a c -vertex or a d -vertex according to the scheme shown in Figure 3. Furthermore, since the vertices of G_m correspond to edges of G , each all-crossing direction of G_m gives rise to a $\{c, d\}$ -labelling of the edges of G . We call the edges of G which correspond to c -vertices of G_m c -edges, and we call the edges of G which correspond to d -vertices of G_m d -edges. (See Figure 2(d).)

We will need the following observation.

Proposition 2. *Let G be a plane graph, then e is a c -edge in G if and only if e^* is a d -edge in G^* .*

Proof. The result follows by observing that the medial graphs G_m and $(G^*)_m$ are equal and that the canonical face 2-colouring of $(G^*)_m$ is obtained from that of G_m by switching the colour of each face. \square

4.2. Bipartite partial duals and medial graphs. We will prove the following characterization of bipartite partial duals in terms of medial graphs.

Theorem 7. *Let G be a plane graph. Then the partial dual G^A is bipartite if and only if A is the set of c -edges arising from an all-crossing direction of G_m .*

We will deduce the theorem from the following lemmas.

Lemma 1. *Let G be a plane graph and let C be a set of c -edges of G arising from an all-crossing direction of G_m . Then each component of $G|_C$ and of $G^*|_{C^c}$ is Eulerian.*

Proof. Consider a vertex v of G , with half-edges e_1, e_2, \dots, e_n adjacent to it. This vertex v corresponds to a face f of G_m , bounded by edges of G_m joined by (not necessarily distinct) vertices w_1, w_2, \dots, w_n corresponding to the half-edges e_1, e_2, \dots, e_n of G . Let us take a position on the half-edge $w_n w_1$ of G_m , and then walk around the face f until we return to our starting point. We note that because G_m has an all-crossing direction, each half-edge will be directed, and we start our walk in this given direction.

We walk until we meet our first vertex, which without loss of generality we may take to be w_1 . This vertex has a label, either c or d . Let us assume that it is a c -vertex; see Figure 4. When we cross w_1 and continue walking on to the next half-edge $w_1 w_2$ we will walk against the given direction of $w_1 w_2$. In order for us to get back to where we started, the half-edges along which we walk will have to change direction in total an even number of times. Therefore there must be an even number of c -vertices around the face f , and so v is adjacent to an even number of c -half-edges.

Now let us assume that the first vertex we meet w_1 is a d -vertex. When we cross this vertex the next half-edge of G_m will be directed compatibly with the direction in which we are walking. So we keep walking until we meet a c -vertex, and then the previous argument applies. If we do not encounter any c -vertices then all the half-edges adjacent to the vertex v are labelled d .

Since a graph is Eulerian if and only if each of its vertices is of even degree, it follows that each component of $G|_C$ is Eulerian. Also, it follows from Proposition 2 that each component of $G^*|_{C^c}$ is Eulerian. \square

(In fact it is not difficult to show the equivalent property that any cycle of G contains an even number of d -edges.)

Lemma 2. *Given a $\{c, d\}$ -colouring of the edges of a plane graph G , with the set of c -edges denoted A , if each component of $G|_A$ is Eulerian and each component of $G^*|_{A^c}$ is Eulerian, then the $\{c, d\}$ -colouring arises from an all-crossing direction of G_m .*

Proof. The graph G is plane, so it is equipped with an embedding $i : G \rightarrow S^2$. This induces an embedding $G^* \rightarrow S^2$, which we also denote i (so that $i(G) \cup i(G^*)$ is the standard immersion). Now take $i(G|_A) \cup i(G^*|_{A^c})$, and denote the resulting graph by Φ .

We first show that the regions of Φ must be either discs or annuli, by arguing that no two components of $G^*|_{A^c}$ can lie in the same region of $i(G|_A)$.

Consider a region f of $i(G|_A)$, drawn in G . If it is a region of G , then it is a disc. If it is not a region of G , then it contains other edges and vertices of G , all the edges being marked d . These edges and vertices divide f into regions of G . But G is connected, so we can walk between any two of these regions via other such regions. Therefore, the part of $G^*|_{A^c}$ lying in f must be connected, as required.

In Φ^\circledast , this means that the deletion of any cut vertex splits Φ^\circledast into exactly two components.

We can also see that Φ^\circledast is bipartite, as follows. Each component of the (disjoint) graphs $G|_A$ and $G^*|_{A^c}$ is Eulerian, and so each vertex of Φ has even degree. Hence each region of Φ^\circledast has an even number of edges. Any cycle in Φ^\circledast can be formed by adding boundaries of regions mod 2. Hence the result.

Note next that because we will be switching between directed graphs and their duals it is natural to consider not only the usual ‘‘longitudinal’’ direction along an edge, but also a ‘‘transverse’’ direction.

Now choose a block B_1 of Φ^\circledast . It is bipartite, and has no cut vertices. Choose an arbitrary transverse direction on an edge e^\circledast in B_1 , and use this to determine a clockwise and anti-clockwise orientation of the vertex at each of the ends of e^\circledast . (See Figure 5(a).) Any other vertex in B_1 will

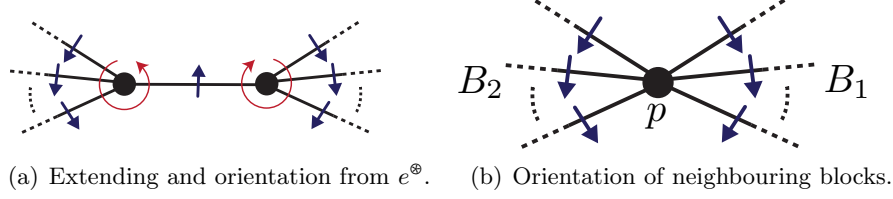


FIGURE 5. Extending an orientation of Φ^{\otimes} .

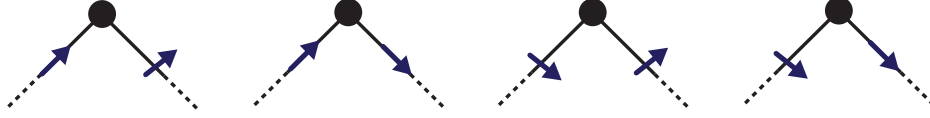


FIGURE 6. Four possible situations for two edges which share both a vertex and a region.

be clockwise if it is an even distance from a clockwise vertex, and anti-clockwise otherwise. This is consistent because B_1 is bipartite, and so it has no odd cycles.

Next, back in Φ^{\otimes} , let p be a cut vertex of B_1 . (If B_1 has no cut vertex, then $B_1 = \Phi^{\otimes}$ and we skip this step.) By our observation above, there is a next block: call it B_2 and orient it as in Figure 5(b). Proceed in this way until all the edges of Φ^{\otimes} have been given a direction.

Finally, transfer this direction back to the edges of Φ . This in turn induces a direction for the edges of G , mixed, in the sense that the d edges are directed transversely.

The vertices of the medial graph G_m inherit their c or d status from the edges of G , and around each vertex the edges have now been directed as in Figure 3. We have to check that as we move from one vertex of G_m to another along an edge, the directions are consistent. To see this, note that for any two edges in G which share both a vertex and a region, our direction must give one of the four situations in Figure 6, or their opposite orientations. So the ‘‘local’’ directions around each vertex of G_m do arise from a ‘‘global’’ direction of the edges of G_m , and we thus have an all-crossing direction, as required. \square

Proof of Theorem 7. By combining Lemmas 1 and 2, we have that each component of $G|_A$ and of $G^*|_{A^c}$ is Eulerian if and only if A is the set c -edges of G arising from an all-crossing direction of G_m . The result then follows by Theorem 6. \square

Corollary 1. *Let G be a plane graph. Then G has at most $2^{c(G_m)-1}$ bipartite partial duals.*

Proof. G_m admits $2^{c(G_m)}$ all-crossing directions, each direction giving rise to a bipartite partial dual by Theorem 7. However, reversing the direction of each edge in an all-crossing direction of G_m does not change the $\{c, d\}$ -colouring of G , accounting for the ‘ -1 ’ in the exponent. There may be a further reduction in the number of bipartite partial duals, as each $\{c, d\}$ -colouring of G need not result in a distinct partial dual of G . \square

The following corollary of Theorem 7 provides a way of constructing some, but not all, of the Eulerian partial duals of a plane graph.

Corollary 2. *Let G be a plane graph. If A is the set of d -edges arising from an all-crossing direction of G_m , then G^A is Eulerian.*

Proof. Since A is the set of d -edges of G , A^c is the set of c -edges of G . Then

$$G^A = G^{(A^c \Delta E(G))} = (G^{A^c})^*.$$

By Theorem 7, G^{A^c} is bipartite, and, since the geometric dual of any bipartite graph is Eulerian, $(G^{A^c})^*$ is Eulerian as required. \square

We note that the converse of Corollary 2 is false and that determining exactly which subsets of edges of a plane graph give rise to Eulerian partial duals remains an open problem.

Remark 3. The proof that the dual of a bipartite embedded graph is Eulerian is almost identical to the well-known proof of the special case for plane graphs: let G be a bipartite graph, then every closed walk in G is of even length (see [1] for example). Therefore every closed walk about a face in any embedding of G is of even length, and it follows that G^* is Eulerian.

Remark 4. The results on partial duals presented in this paper are intimately related to, and motivated by, the authors' work with N. Virdee on the ribbon graphs of link diagrams in [5]. In the context of knot theory, bipartite embedded graphs arise as the Seifert graphs of an oriented link diagram, and these embedded graphs are necessarily partial duals of plane graphs. This provides a link between the graph theory presented in Section 4 and knot theory. In fact, our knot theoretic results from [5] on the characterization of Seifert graphs suggested the formulation and proof of Theorem 7 to us. Furthermore, the connection between the Tait graph and Seifert graph of a link diagram that was also studied in [5], led the authors to conjecture that $[(G \cup G^*) - (A^c \cup A^*)]^\circledast$ and G^A are isomorphic as abstract (but not necessarily embedded) graphs, which appears as Theorem 5 below. The results presented here therefore further illustrate the deep and fruitful connections between knot theory and graph theory.

5. THE PROOF OF THEOREM 5

We will prove Theorem 5 by using the characterization of partially dual graphs in terms of a bijection between edge sets from [10]. This extends the usual characterization of dual graphs in terms of maps between edge sets, which is due to Whitney for plane graphs [15], and Edmonds for higher genus graphs [3]. We will need to introduce a little notation.

Suppose that G and H are graphs and $\varphi : E(G) \rightarrow E(H)$ is a bijection between their edge sets. Let $v \in V(G)$ and $S \subseteq E(G)$. Then we let S_v be the set of edges in S which are incident with v . Then the set $\varphi(S_v)$ of edges in H together with the vertices which are incident with these edges form a subgraph of H . This subgraph is denoted by $\varphi(S)_v$. Furthermore, we let H_v denote the subgraph $\varphi(E(G))_v$.

Definition 5. Let G and H be graphs and $\varphi : E(G) \rightarrow E(H)$ be a bijection. We say that φ satisfies *Edmonds' Criteria* if

- (1) edges $e, f \in E(G)$ belong to the same connected component if and only if $\varphi(e), \varphi(f) \in E(H)$ belong to the same connected component;
- (2) for each $v \in V(G)$, each component of H_v is Eulerian;
- (3) for each $v \in V(H)$, each component of G_v is Eulerian.

The significance of Edmonds' Criteria is that they provide a characterization of geometric duality in terms of mappings between edge sets. Edmonds showed in [3] that $H \cong G^*$ for some cellular embedding of a graph G , if and only if there exists a bijection $\varphi : E(G) \rightarrow E(H)$ that satisfies Edmonds' Criteria and that G and H have the same number of isolated vertices. Edmonds' Theorem is an extension of Whitney's characterization of planar duals in terms of combinatorial duals (see [15]) to graphs embedded in an arbitrary surface.

In [10], the second author extended Edmonds' and Whitney's Theorems to partial duals:

Theorem 8. *Two graphs G and H are partial duals if and only if there exists a bijection $\varphi : E(G) \rightarrow E(H)$, such that*

- (1) $\varphi|_A : A \rightarrow \varphi(A)$ satisfies *Edmonds' Criteria* for some subset $A \subseteq E(G)$;

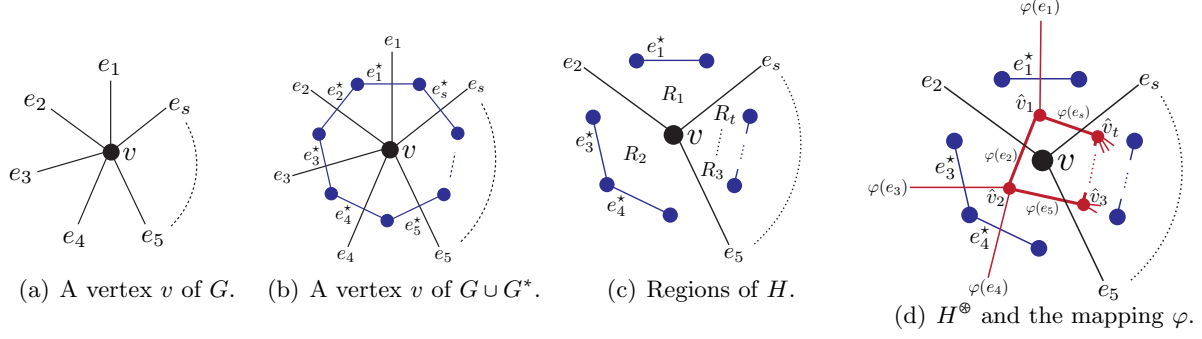


FIGURE 7. Figures used in the proof of Theorem 5.

- (2) If $v \in V(G)$ is incident with an edge in A , and if $e \in E(G)$ is incident with v , then $\varphi(e)$ is incident with a vertex of $\varphi(A)_v$. Moreover, if both ends of e are incident with v , then both ends of $\varphi(e)$ are incident with vertices of $\varphi(A)_v$.
- (3) If $v \in V(G)$ is not incident with an edge in A , then there exists a vertex $v' \in V(H)$ with the property that $e \in E(G)$ is incident with v if and only if $\varphi(e) \in E(H)$ is incident with v' . Moreover, both ends of e are incident with v if and only if both ends of $\varphi(e)$ are incident with v' .

Furthermore, with A as above $H \cong G^A$.

We recall that $\varphi(A)_v$ is the subgraph of H induced by the images of the edges from A that are incident with v .

While the claim in the above theorem that $H \cong G^A$ did not appear explicitly in [10], it is an immediate consequence of the proof of Theorem 26 of [10]. We will use this characterization of partial duals to prove Theorem 5.

Proof of Theorem 5. To avoid clutter in the proof, we set $H := (G \cup G^*) - (A^c \cup A^*)$. We begin by defining a mapping $\varphi : E(G) \rightarrow E(H^\circled)$, and go on to show that this mapping satisfies the conditions of Theorem 8.

Each edge $e \in E(G)$ is naturally identified with exactly one edge e^* in $E(G^*)$, and exactly one of e or e^* is in H . This gives a natural bijection α between $E(G)$ and $E(H)$. If we let β denote the natural bijection between $E(H)$ and $E(H^\circled)$, then we obtain a natural bijection between $\varphi : E(G) \rightarrow E(H^\circled)$ by setting $\varphi := \beta \circ \alpha$.

To prove the theorem it remains to show that φ satisfies the conditions of Theorem 8 with the edge set A given in the Theorem. For the first condition, we note that $A = E(G - A^c)$ and so, by the definition of α , we have $\alpha(A) = E(G - A^c)$. Then, since β is just the natural identification of edge sets of a graph and its dual, we have

$$\varphi(A) = \beta(E(G - A^c)) = E((G - A^c)^\circled),$$

so

$$\varphi|_A : E(G - A^c) \rightarrow E((G - A^c)^\circled).$$

It is then readily verified that $\varphi|_A$ satisfies Edmonds' Criteria.

We will now show that φ satisfies the remaining conditions of Theorem 8. To do this we consider the construction of H^\circled from the embedded graph G locally in the neighbourhood of a vertex v of G . We begin with the immersed graph $G \cup G^*$. Let $v \in V(G)$, and let e_1, e_2, \dots, e_s be the (not necessarily distinct) cyclically ordered edges of G that are incident to v , where the cyclic order is chosen with respect to an arbitrary orientation of a neighbourhood of v . See Figure 7(a). Let e_i^* denote the unique edge in the subgraph G^* of $G \cup G^*$ which intersects e_i , for each i . See Figure 7(b).

Note that the e_i^* s which arise need not be distinct. We will let D_v denote the s -gon, together with the embedding of v and its incident half-edges, which is obtained by cutting the immersed graph $G \cup G^*$ along the edges e_1^*, \dots, e_s^* and their incident vertices. Next form H by deleting all of the edges of the immersed graph $G \cup G^*$ which belong to $A^c \cup A^*$. The remaining edges of G which are incident with v divide D_v into regions R_1, \dots, R_k in the following way: if no edges in A are incident with v then there is a single region R_1 ; if there are edges in A that are incident with v , then cyclically order the regions R_1, R_2, \dots, R_t according to some orientation of D_v . See Figure 7(c). Note that the regions R_1, R_2, \dots, R_t need not be distinct regions of the embedded graph H . Finally, form H^\otimes . There is a vertex \hat{v}_k of H^\otimes associated with each region R_k (but the vertices $\hat{v}_1, \dots, \hat{v}_t$ need not be distinct). See Figure 7(d). In addition, notice that:

- if the edge e_i of H is adjacent to the region R_k , then $\varphi(e_i)$ is the edge which is incident with \hat{v}_k and which intersects e_i in the canonical immersion of $H \cup H^\otimes$;
- if the edge e_i^* in H is adjacent to the region R_k , then $\varphi(e_i)$ is the edge which is incident with \hat{v}_k and which intersects e_i^* in the canonical immersion of $H \cup H^\otimes$;
- if v is incident with an edge in A , then every \hat{v}_k is in $\varphi(A)_v$.

We will use these observations (which are illustrated in Figure 7(d)) to verify that the map $\varphi : E(G) \rightarrow E(H^\otimes)$ satisfies the remaining conditions of Theorem 8. There are three cases to consider.

Case 1: Suppose that v is incident with an edge in A and $e \in A$. In this case, by definition, $\varphi(e) \in \varphi(A)_v$ and so both ends of $\varphi(e)$ are in $\varphi(A)_v$.

Case 2: Suppose that v is incident with an edge in A and $e \notin A$. In this case, we have that $\varphi(e) = e_i^*$, for some i . Then e_i^* is adjacent to a region R_k , and hence $\varphi(e)$ is incident with \hat{v}_k , and therefore incident with a vertex of $\varphi(A)_v$.

If e is a loop then $\varphi(e) = e_i^* = e_j^*$, for some i and j . Then e_i^* is adjacent to a region R_k , and e_j^* is adjacent to a region R_l . This means that $\varphi(e)$ is incident with \hat{v}_k and with \hat{v}_l (with incidence being counted twice if $\hat{v}_k = \hat{v}_l$). Thus, both ends of $\varphi(e)$ are incident with $\varphi(A)_v$.

Case 3: Suppose that v is not incident with an edge in A . In this case there is only one region R_1 , and the vertex \hat{v}_1 is the vertex v' required by Theorem 8. To see why this is the case, suppose that $e \in E(G)$ is incident with v . Then $\varphi(e) = e_i^*$, for some i , and, as before, e_i^* is adjacent to a region R_1 , so $\varphi(e)$ is incident with \hat{v}_1 , as required. If, in addition, e is a loop, then $\varphi(e) = e_i^* = e_j^*$, for some i and j . Then since e_i^* and e_j^* are both adjacent to R_1 both ends of $\varphi(e)$ are incident to \hat{v}_1 .

Thus we have shown that φ satisfies the conditions of Theorem 8, and so $H \cong G^A$ as required. \square

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