

EVALUATIONS OF TOPOLOGICAL TUTTE POLYNOMIALS

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ABSTRACT. We find a number of new combinatorial identities for, and interpretations of evaluations of, the topological Tutte polynomials of Las Vergnas, $L(G)$, and of and Bollobás and Riordan, $R(G)$, as well as for the classical Tutte polynomial $T(G)$. For example, we express $R(G)$ and $T(G)$ as a sum of chromatic polynomials, show that $R(G)$ counts non-crossing graph states and k -valuations, and reformulate the Four Colour Theorem in terms of $R(G)$. Our main approach is to apply identities for the topological transition polynomial, one involving twisted duals, and one involving doubling the edges of a graph. These identities for the transition polynomial allow us to show that the Penrose polynomial $P(G)$ can be recovered from $R(G)$, a fact that we use to obtain identities and interpretations for $R(G)$. We also consider enumeration of circuits in medial graphs and use this to relate $R(G)$ and $L(G)$ for graphs embedded in low genus surfaces.

1. INTRODUCTION

Although graph polynomials are well studied, research in this area has traditionally focused on polynomials of abstract graphs. Recently, however, there has been a growing interest in topological graph polynomials, that is, polynomials of graphs embedded in surfaces. This interest is coming not only from graph theory, but also from other areas such as knot theory and quantum field theory. Here we focus on topological Tutte polynomials, that is, any polynomial of embedded graphs from which the classical Tutte polynomial may be recovered. Las Vergnas introduced a topological Tutte polynomial, $L(G)$, in 1978 (see [23, 24, 26]), but most of the recent interest in this area has been spurred by a (different) topological Tutte polynomial, Bollobás and Riordan’s ribbon graph polynomial, $R(G)$, of [6, 7]. Remarkably little is yet known about what information is encoded by topological Tutte polynomials.

Very often an understanding of one graph polynomial leads to new results for graph polynomials related to it. Here we have two powerful identities for the topological transition polynomial, $Q(G)$, introduced in [15]: one for any twisted dual of a graph, and one for the “double” of a graph, that is, the result of replacing each of its edges by two edges in parallel. Both twisted duality and the double operation act in a geometric way on an embedded graph. On the algebraic side, these two operations act on the transition polynomial in a way that is compatible with the geometric action. These actions therefore provide a way to move between the structure of an embedded graph and the structure of graph polynomials. Because $Q(G)$ specializes to $P(G)$, an extension of the Penrose polynomial to embedded graphs given in [16], and agrees with the topochromatic polynomial (which is a multivariable translate of $R(G)$) $Z(G)$ for certain classes of evaluations, these actions allow us to translate the new identities and evaluations for $Q(G)$ into new results for the various topological Tutte polynomials. Several of these results specialize to new identities for the classical Tutte polynomial. Among the results of this paper are:

- a restatement of the Four Colour Theorem in terms of $R(G)$;
- expressions of $R(G)$ and the classical Tutte polynomial as sums of chromatic polynomials;
- that $R(G)$ specializes to the topological Penrose polynomial $P(G)$ via taking the geometric dual and the Petrie dual of an embedded graph;
- a relation between $R(G)$ and $L(G)$ for graphs embedded in low genus surfaces,
- a generalization of the partial duality relation for $R(G)$ to $Z(G)$;

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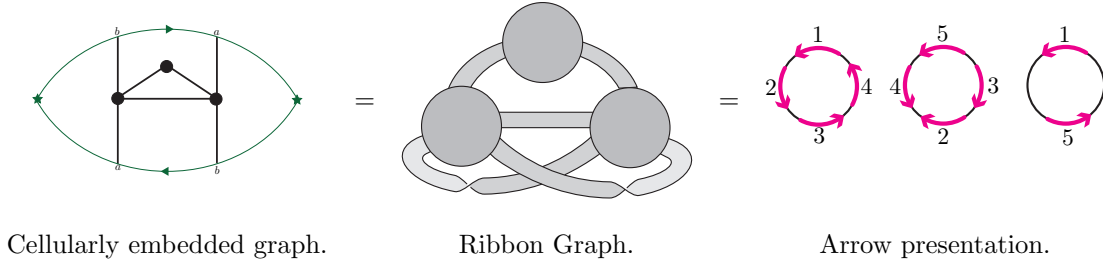


FIGURE 1. Three descriptions of the same embedded graph.

- that $R(G; t + 1, t, \frac{1}{t}, 1)$ is a generating function for non-crossing graph states in its embedded medial graph G_m ;
- that $R(G)$ counts the number of special edge colourings in graphs associated with G ; and
- a variety of evaluations of $R(G)$ whose expressions involve not only combinatorial, but also topological, information about G .

The tools for achieving the results above are twisted duality, the ribbon group, and a variety of different embedded graph polynomials. Sections 2, 3, and 4, provide the necessary background. Readers familiar with [6, 7, 15, 16] and the associated theory may safely skip these sections, except for Subsection 3.3 which describes $L(G)$ in graph theoretical terms, and Subsection 4.3 which gives the new doubled graph identity for the topological transition polynomial.

The main results, including those in the list above, appear in Sections 5 and 6. Section 5 relates all the various polynomials, and translates identities among them. Section 6 gives further identities and many new evaluations. In Subsection 6.4 we show that the ribbon group action on the transition polynomial provides a framework for understanding and uniting recent duality results for $Z(G)$ and $R(G)$ that have been studied by a variety of authors (see [6, 9, 18, 29, 33]). Finally, in Subsection 6.5 we use the tools developed here to extend some low genus enumeration results of Las Vergnas from [24] to all surfaces.

2. EMBEDDED GRAPHS

This section provides a brief overview of results from [15], and we refer the reader to this reference for further details of the results described in this section. These results form the foundation of the rest of this paper.

2.1. Descriptions of embedded graphs. We use the term “embedded graph” loosely to mean any of several equivalent representations of graphs in surfaces. We may think of an embedded graph as any of:

- (1) a cellularly embedded graph, that is, a graph embedded in a surface such that every face is a 2-cell;
- (2) a ribbon graph, also known as a band decomposition;
- (3) an arrow presentation, or signed rotation system.

We will use these equivalent representations interchangeably, choosing whichever best facilitates the discussion at hand. (See Figure 1.) We assume the reader is familiar with cellular embeddings of graphs, ribbon graphs and with their equivalence, but we briefly review Chmutov’s arrow presentations as they are more recent.

Definition 2.1 (Chmutov [9]). An *arrow presentation* consists of a set of circles, each with a collection of disjoint, labelled arrows, called *marking arrows*, lying on them. Each label appears on precisely two arrows.

A ribbon graph can be obtained from an arrow presentation as follows. Vertices are formed by viewing each circle as the boundary of a disc that becomes a vertex of the ribbon graph. Edges are then added to the vertex discs in the following way. Take an oriented disc for each label of the marking arrows. Choose two non-intersecting arcs on the boundary of each of the edge discs and direct these according to the orientation. Identify these two arcs with two marking arrows, both with the same label, aligning the direction of each

arc consistently with the orientation of the marking arrow. This process can be illustrated pictorially thus:



Conversely, to describe a ribbon graph G as an arrow presentation, start by arbitrarily labelling and orienting the boundary of each edge disc of G . On the arc where an edge disc intersects a vertex disc, place an arrow on the intersection, labelling the arrow with the label of the edge it meets and directing it consistently with the orientation of the edge disc boundary. The boundaries of the vertex set marked with these labelled arrows give the arrow marked circles of an arrow presentation. See Figure 1 for an example, and [9] for further details.

Two cellularly embedded graphs are *equivalent* if there is a homeomorphism of the ambient space taking one graph into the other. Ribbon graphs or arrow presentations are equivalent if they describe equivalent cellularly embedded graphs.

2.2. Some notation. We use standard notation: $V(G)$ and $E(G)$ denote the vertex set and edge set, respectively, of an embedded graph G , while $v(G)$ and $e(G)$, respectively, are the numbers of vertices and edges. We let $k(G)$, $r(G)$, and $n(G)$, respectively, denote the number of components, rank, and nullity of the underlying abstract graph. If G is viewed as a ribbon graph, then $f(G)$ is the number of boundary components of the surface defining the ribbon graph, and if G is viewed as a cellularly embedded graph, $f(G)$ is the number of its faces.

An embedded graph is *orientable* if it is cellularly embedded in an orientable surface (or equivalently, the ribbon graph is an orientable surface), and is *non-orientable* otherwise. The function t records the orientability of an embedded graph G by $t(G) = 0$ if G is orientable and $t(G) = 1$ otherwise.

The *genus* of a connected embedded graph is the genus of the surface it is embedded in when realized as a cellularly embedded graph. The genus of an embedded graph is defined as the sum of the genera of its components. We define

$$\gamma(G) := \sum_{\substack{C_i \\ \text{orient.}}} 2g(C_i) + \sum_{\substack{C_i \\ \text{non-orient.}}} g(C_i),$$

where the sums are over the orientable and non-orientable components $C_1, \dots, C_{k(G)}$ of G , respectively, and $g(C_i)$ is the genus of C_i .

The notion of a spanning ribbon subgraph is clear, and we define a spanning subgraph of an embedded graph to be the embedded graph (in any of our equivalent formulations of embedded graphs) corresponding to a spanning ribbon subgraph. (This definition ensures that the spanning subgraphs of a cellularly embedded graph are also cellularly embedded, although not necessarily in the same surface as the original graph.) If G is an embedded graph and $A \subseteq E(G)$, then we will often use $G|_A$ to denote the embedded spanning subgraph $(V(G), A)$ of G . We let $r(A)$, $k(A)$, $n(A)$, $f(A)$, $t(A)$, and $\gamma(A)$ each refer to the spanning subgraph of G on the edge set A , with embedding inherited from G . In cases where we need to specify the graph G , we will write $r_G(A)$, $k_G(A)$, etc..

If G^* is the geometric dual of an embedded graph G , then there is a natural bijection between $E(G)$ and $E(G^*)$. Thus we can, and will, identify the edges of G and G^* . Therefore, for every spanning subgraph of $G|_A$ of G , there is a corresponding spanning subgraph of $G^*|_A$ of G^* . Using this correspondence we establish the notation that if $A \subseteq E(G)$, then $r_{G^*}(A)$ is the rank of $G^*|_A$, and make similar definitions for $k_{G^*}(A)$, $n_{G^*}(A)$, $f_{G^*}(A)$, $t_{G^*}(A)$, and $\gamma_{G^*}(A)$.

2.3. Medial graphs. Many of the interpretations of topological Tutte polynomials and the interrelations among the various graph polynomials presented in this paper rely upon medial graphs. If a graph G is cellularly embedded, its *medial graph*, G_m , is constructed by placing a vertex of degree 4 on each edge of G , and then drawing the edges of the medial graph by following the face boundaries of G . Consistent with this definition is that the medial graph of an isolated vertex is an isolated face, and we adopt this convention. Note that if G is cellularly embedded in a surface Σ , then G_m is also cellularly embedded in Σ .

A *checkerboard colouring* (or face 2-colouring) of a $2k$ -regular embedded graph F is an assignment of the colour black or white to each of its faces such that adjacent faces have different colours. While not all $2k$ -regular embedded graphs admit checkerboard colourings, all medial graphs do. In fact, we can associate

a canonical checkerboard colouring to a medial graph as follows. In the construction of G_m , the vertices of G appear in some of the faces of G_m . Thus we can associate a face of G_m with each vertex of G . We can then construct a checkerboard colouring of G_m by colouring a face black if it is associated with a vertex of G , and colouring it white otherwise. Such a checkerboard colouring is called the *canonical checkerboard colouring* of G_m .

2.4. Weight systems and graph states. We will need weight systems to define the transition and Penrose polynomials in the next section, and they will be key tools in our study of graph polynomials presented here. While weight systems are more generally defined (see [17]), we will only need to apply them to 4-regular embedded graphs and so we only consider this case.

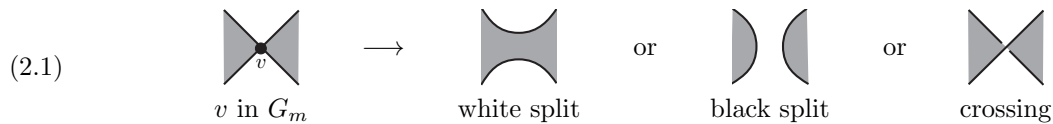
A *vertex state* at a vertex v of an abstract 4-regular graph F is a partition into pairs of the edges incident with v . Thus, if F is an cellularly embedded graph, a vertex state is simply a choice of one of the following configurations in a neighbourhood of the vertex v :



The configurations replace a small neighbourhood of the vertex v . We will refer to the first two of the these vertex states as *splits* and the third as a *crossing*.

A *graph state* s of F is a choice of vertex state at each vertex of F .

If G is an embedded graph and G_m its canonically checkerboard coloured medial graph, then we can use the checkerboard colouring to distinguish among the vertex states at a given vertex. We will name of each type of vertex state as being either a *white split*, *black split* or a *crossing* as defined in the figure below.



We denote the set of graph states of a 4-regular graph F by $St(F)$. If G_m is the canonically checkerboard coloured medial graph of G , then $\mathcal{P}(G_m)$ denotes the set of graph states with no black splits, and we will call such states *Penrose states*; and $\mathcal{Z}(G_m)$ denotes the set of graph states with no crossings, and we will call such states *Tutte states*.

A *weight system*, $W(F)$, of any 4-regular graph F (embedded or not) is an assignment of a weight in a unitary ring \mathcal{R} to every vertex state of F . (We simply write W for $W(F)$ when the graph is clear from context.) If s is a state of F , then the *state weight* of s is $\omega(s) := \prod_{v \in V(F)} \omega(v, s)$, where $\omega(v, s)$ is the vertex state weight of the vertex state at v in the graphs state s . Note that a state s consists of a set of disjoint closed curves, and we refer to these as the *components of the state*, denoting the number of them by $c(s)$.

If G_m is an embedded medial graph, then we can construct a weight system by associating a weight to white splits, black splits and crossings as in the definition below.

Definition 2.2. Let G be an embedded graph with embedded medial graph G_m . Define the *medial weight system*, $W_m(G_m)$, using the canonical checkerboard colouring of G_m as follows. A vertex v has state weights given by an ordered triple $(\alpha_v, \beta_v, \gamma_v)$, indicating the weights of the white split, black split, and crossing state, in that order. We write $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ for the set of these ordered triples, indexed, equivalently, either by the vertices of G_m , or by the edges of G .

Graphically, the medial weight system $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ will assign weights to a vertex v of G_m as follows:

state			
weight	α_v	β_v	γ_v

We will use the following convention: if in a medial weight system $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma})$ we have $\boldsymbol{\alpha} = \mathbf{k}$, where \mathbf{k} is in the ring \mathcal{R} , we mean that each $\alpha_v = k$, and similarly for $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$. For example, $(\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}) = (-\mathbf{1}, \mathbf{0}, \mathbf{1})$, denotes the medial weight system where $\alpha_v = -1$, $\beta_v = 0$, and $\gamma_v = 1$, for each v .

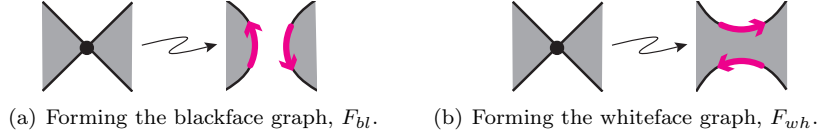
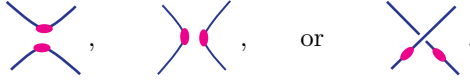


FIGURE 2. Forming the Tait graphs of F .

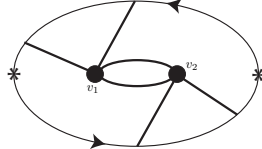
2.5. Cycle family graphs. Cycle family graphs were introduced in [15] to provide a way to recover every embedded graph that has a given 4-regular *abstract* graph as its medial graph. Cycle family graphs extend the well-known Tait graphs, which provide a way to recover every embedded graph that has a given 4-regular *embedded* graph as its medial graph.

Let F be a 4-regular embedded graph. An *arrow marked vertex state* of a vertex v in F consists of a vertex state equipped with exactly two v -labelled arrows. Each arrow is placed on one of the positions indicated below and may point in either direction.

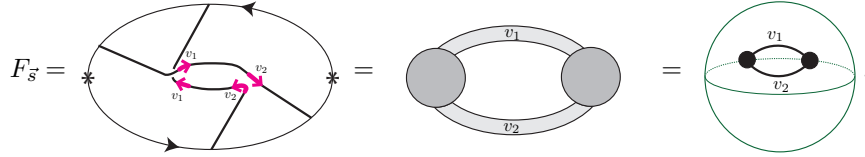


An *arrow marked graph state* \vec{s} of F is a choice of arrow marked vertex state at each vertex of F .

If \vec{s} is an arrow marked graph state of F , we can regard \vec{s} as an arrow presentation of an embedded graph by viewing each component of \vec{s} as a circle with marking arrows. We denote this embedded graph by $F_{\vec{s}}$ and call it a *cycle family graph* of F . We denote the set of all cycle family graphs of a 4-regular embedded graph F by $\mathcal{C}(F)$. Note that there is a natural identification between the vertex set of F and the edge set of $F_{\vec{s}}$.

Example 2.3. If $F = *$  $*$, is a graph embedded in the projective plane, then one of the

arrow marked graph state \vec{s} of F gives the following cycle family graph:



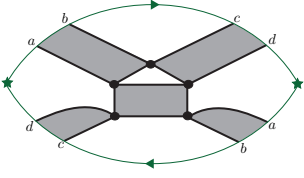
We will need the following theorem.

Theorem 2.4 ([15]). *If F is any 4-regular graph, then the set of cycle family graphs of any embedding of F is precisely the set of all embedded graphs G such that G_m and F are equivalent as abstract graphs, i.e. if \tilde{F} is any embedding of F , then*

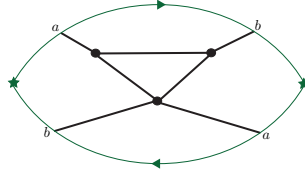
$$\mathcal{C}(\tilde{F}) = \{G | G_m \cong \tilde{F}\} = \{G | G_m \cong F\}.$$

It was shown in [15] that, in the case when F is checkerboard colourable, the two Tait graphs F_{bl} and F_{wh} of F arise as cycle family graphs of F . These can be constructed as follows. Let F be a checkerboard coloured, 4-regular embedded graph. The *blackface graph*, F_{bl} , of F is the cycle family graph constructed by choosing the vertex state shown in Figure 2(a) at each vertex; and the *whiteface graph*, F_{wh} , of F is the cycle family graph constructed by choosing the vertex state shown in Figure 2(b) at each vertex.

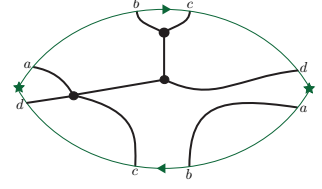
Example 2.5. An example of a checkerboard coloured graph embedded on the projective plane and its blackface and whiteface graphs is shown below.



A 4-regular checkerboard coloured embedded graph F .



Its blackface graph F_{bt} .



Its whiteface graph F_{wh} .

2.6. Twisted duals and the ribbon group. Twisted duals and the ribbon group action, which generates them, were both introduced in [15]. Twisted duality is a far reaching generalization of the geometric dual of an embedded graph that subsumes both Petrie duality and Chmutov's partial duality of [9]. The importance of the ribbon group lies in that it provides a new understanding of the relationship between graphs and their medial graphs. In this subsection we provide a brief overview of twisted duality and the ribbon group action. We will take the most direct approach in our descriptions, avoiding a full discussion of group actions, referring the reader to [15] for details.

Definition 2.6. Let G be an embedded graph described as an arrow presentation, and let e be an edge of G . The *half-twist of the edge e* , denoted $\tau(e)$, is the operation that reverses the direction of exactly one of the e -labelled arrows of the arrow presentation. We let $G^{\tau(e)}$ denote the embedded graph obtained by applying $\tau(e)$ to G .

The *partial dual with respect to the edge e* , denoted $\delta(e)$, is the operation that changes the arrow presentation as follows. Suppose A and B are two arrows labelled e in an arrow presentation of G . Draw a line segment with an arrow on it directed from the the head of A to the tail of B , and a line segment with an arrow on it directed from the head of B to the tail of A . Label both of these arrows e , then delete A and B and the arcs containing them. We let $G^{\delta(e)}$ denote the embedded graph obtained by applying $\delta(e)$ to G .

The actions of $\tau(e)$ and $\delta(e)$ on an arrow presentation (where all arrows in the figure have the same label e) are illustrated thus:

$$\tau \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \begin{array}{c} \curvearrowleft \\ \curvearrowright \end{array} \quad \text{and} \quad \delta \left(\begin{array}{c} \curvearrowright \\ \curvearrowleft \end{array} \right) = \begin{array}{c} \curvearrowright \\ \curvearrowright \end{array}.$$

For $\xi, \zeta \in \{\delta, \tau\}$, by setting $\xi\zeta(e) := \zeta(e)\xi(e)$ we obtain an action of the free group $\langle \delta, \tau \rangle$ on a set of embedded graph with a distinguished edge e . For example, we have $G^{\tau\tau\delta(e)} = ((G^{\delta(e)})^{\tau(e)})^{\tau(e)}$.

It was shown in [15] that, for each edge e , $G^{\tau^2(e)} = G^{\delta^2(e)} = G^{(\delta\tau)^3(e)} = G$, and so the group

$$\mathfrak{G} := \langle \delta, \tau \mid \delta^2, \tau^2, (\tau\delta)^3 \rangle,$$

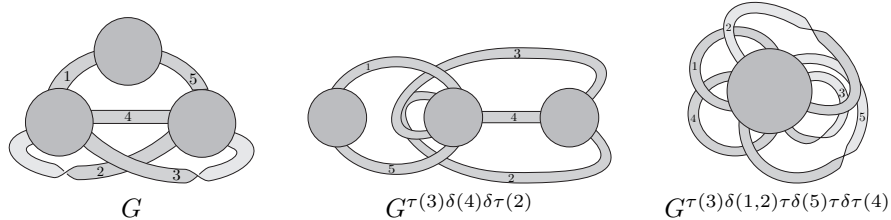
acts on embedded graphs with a distinguished edge e . Note that \mathfrak{G} is isomorphic to the symmetric group of order three.

If $A \subseteq E(G)$ and $\xi \in \mathfrak{G}$, we define $G^{\xi(A)}$ to be the embedded graph obtained by applying ξ to every edge in A . It was shown in [15] that $G^{\xi(A)}$ is independent of the order in which we apply ξ to the edges in A . If ζ is also in \mathfrak{G} it follows that $G^{\xi\zeta(A)} = G^{\zeta(A)\xi(A)}$ and that $G^{\xi(A)\zeta(B)} = (G^{\xi(A)})^{\zeta(B)}$. An important observation is that if A and B are disjoint, then $\zeta(A)\xi(B) = \xi(B)\zeta(A)$ (see [15]).

Definition 2.7. Let G and H be embedded graphs. Then G and H are *twisted duals* if $H = G^{\prod_{i=1}^k \xi_i(A_i)}$ for some $A_i \subseteq E(G)$ and $\xi_i \in \mathfrak{G}$, for $i = 1, \dots, k$.

With this notation, it can be shown (see Proposition 3.7 of [15]) that every twisted dual of an embedded graph G admits an expression of the form $G^{\prod_{i=1}^6 \xi_i(A_i)}$, where the A_i 's partition $E(G)$, and where $\xi_1 = 1, \xi_2 = \tau, \xi_3 = \delta, \xi_4 = \tau\delta, \xi_5 = \delta\tau$, and $\xi_6 = \tau\delta\tau \in \mathfrak{G}$.

Example 2.8. An embedded graph and two of its twisted duals are shown below.



Other important forms of twisted duality appear as actions of subgroups of the ribbon group on the set of embedded graphs.

Definition 2.9. Let G be an embedded graph and $A \subseteq E(G)$. Then

- (1) $G^* := G^{\delta(E(G))}$ is the *geometric dual* of G ,
- (2) $G^\times := G^{\tau(E(G))}$ is the *petrial* or *Petrie dual* of G ,
- (3) $G^{\delta(A)}$ is the *partial dual* of G with respect to A ,
- (4) $G^{\tau(A)}$ is the *partial petrial* of G with respect to A .

In later sections, we will be interested in each of these special cases of twisted duals.

Remark 2.10. As noted above, the classical connections among geometric duals, embedded medial graphs and Tait graphs extends to analogous connections among twisted duals, abstract medial graphs and cycle family graphs. For example, the classical property that $\{(G_m)_{bl}, (G_m)_{wh}\} = \{G, G^*\}$, becomes $\mathcal{C}(G_m) = \text{Orb}(G)$, that is, the cycle family graphs of G_m are precisely the twisted duals of G . (Here $\text{Orb}(G)$ denotes the orbit of G under the ribbon group action, that is, the set of twisted duals of G .) Also, the identity $\{G, G^*\} = \{H : H_m = G_m\}$ becomes $\text{Orb}(G) = \{H : H_m \cong G_m\}$. Thus, just as Tait graph provide the set of graphs with equal embedded medial graphs, twisted duals provide a complete characterization of embedded graphs that have medial graphs that are isomorphic as abstract graphs. Again we refer the reader to [15] for full details of these results.

3. DEFINITIONS AND PROPERTIES OF EMBEDDED GRAPH POLYNOMIALS

Most of the best-studied graph polynomials apply either to plane graphs or to abstract graphs. However, there has been recent interest from mathematicians and physicists in polynomials of embedded graphs. Here we are interested in polynomials of embedded graphs that arise as extensions of classical polynomials for abstract and plane graphs: the topological Tutte polynomials of Las Vergnas [23, 24, 26]) and of Bollobás and Riordan [6, 7], the topological Penrose polynomial [16], and the topological transition polynomial [15].

3.1. The topological Penrose polynomial. The Penrose polynomial was originally defined implicitly by Penrose in [31] for plane graphs and was extended to all embedded graphs by the authors in [16]. By extending the polynomial to all embedded graphs, the authors were able to find a host of new properties of the Penrose polynomial that can not be realized exclusively in terms of plane graphs. Several evaluations of the Penrose polynomial are known to have desirable properties (see for example [1, 16, 31]), and this fact, together with connections to $Z(G)$ given in Section 5, will lead to several of the results of this paper.

Definition 3.1 ([16]). Let G be a plane graph with canonically checkerboard coloured medial graph G_m , let $St(G_m)$ be the set of states of G_m , and let $\mathcal{P}(G_m)$ be its set of Penrose states. Then the *Penrose polynomial* may be defined by

$$P(G; \lambda) := \sum_{s \in St(G_m)} \omega_P(s) \lambda^{c(s)} = \sum_{s \in \mathcal{P}(G_m)} (-1)^{cr(s)} \lambda^{c(s)} \in \mathbb{Z}[\lambda],$$

where $W_P(G_m)$ is the the medial weight system with $\alpha_v = 1$, $\beta_v = 0$, and $\gamma_v = -1$ for all $v \in V(G_m)$, $\omega_P(s)$ denotes its graph state weights, $c(s)$ is the number of components in the graph state s , and $cr(s)$ is the number of crossing vertex states in the graph state s .

The Penrose polynomial may also be computed via a linear recursion relation by repeated applying the skein relation

$$(3.1) \quad \begin{array}{c} \text{X} \\ \text{---} \end{array} = \begin{array}{c} \text{C} \\ \text{---} \end{array} - \begin{array}{c} \text{X} \\ \text{---} \end{array},$$

7

to vertices of degree 4 in G_m , and at the end, evaluating each of the resulting cycles to λ .

The Penrose polynomial turns out to have a number of remarkable graph theoretical properties. Here we need the following evaluations of the Penrose polynomial. Item (1) is due to Penrose [31], and Items (2)-(4) are from [16].

Theorem 3.2. *Let G be an embedded graph.*

(1) *If G is a plane, cubic and connected, then*

$$(3.2) \quad P(G; 3) = (-1/4)^{\frac{v(G)}{2}} P(G; -2) = \text{the number of edge 3-colourings of } G.$$

(2) *If G plane, then*

$$P(G; \lambda) = \sum_{A \subseteq E(G)} \chi((G^{\tau(A)})^*; \lambda).$$

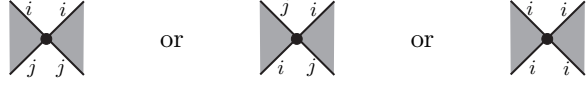
(3) *If G is orientable and checkerboard colourable, then $P(G; 2) = 2^{v(G)}$;*

(4) *If G is orientable and checkerboard colourable, then $P(G; -1) = (-1)^{f(G)} 2^{e(G)}$.*

Note that in Items 3 and 4 of Theorem 3.2, G need not be 4-regular.

We will need the following interpretation of the Penrose polynomial in terms of admissible k -valuations. If $G = (V(G), E(G))$ is an embedded graph and G_m is its embedded medial graph, then a k -valuation of G_m is a edge k -colouring, $\phi : E(G_M) \rightarrow \{1, 2, \dots, k\}$, such that for each i and every vertex v_e of G_m , the number of i -coloured edges incident with v_e is even.

A k -valuation is said to be *permissible* if, at each vertex, the k -valuation is of one of the following three types:



where $i \neq j$. Vertices of the above types in the k -valuation are called *white split*, *crossing* and *total* respectively. We let $\text{Per}(G, k)$ denote the set of permissible k -valuations of G_m . A k -valuation is said to be *admissible* if each vertex in the k -valuation is either a white split or a crossing. We let $\text{Adm}(G, k)$ denote the set of admissible k -valuations of G_m .

The following theorem says that the Penrose polynomial counts k -valuations. Items 1 and 3 are from [16], and Item 2 is due to Jaeger [20] (see also [1] Proposition 4) and also follows from Item 1 by the Jordan Curve Theorem.

Theorem 3.3. *Let G be an embedded graph. Then, for each $k \in \mathbb{N}$,*

(1) *if $\text{cr}(\sigma)$ denotes the number of crossing states in σ ,*

$$P(G; k) = \sum_{\sigma \in \text{Adm}(G, k)} (-1)^{\text{cr}(\sigma)};$$

(2) *if G is plane,*

$$P(G; k) = |\text{Adm}(G, k)|;$$

(3) *if G is orientable and checkerboard colourable, then*

$$P(G; -k) = (-1)^{f(G)} \sum_{\sigma \in \text{Per}(G, k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

3.2. Bollobás and Riordan's topological Tutte polynomial, the topochromatic polynomial, and signed variations. In [6] and [7], Bollobás and Riordan extended the classical Tutte polynomial to embedded graphs with the *ribbon graph polynomial*, $R(G)$, now also known as the *Bollobás-Riordan polynomial*. In [6], they defined the cyclic graph polynomial, a three variable contraction-deletion polynomial for graphs embedded in orientable surfaces. They furthered this work, using a different approach, in [7], with the ribbon graph polynomial $R(G; x, y, z, w)$, a four variable polynomial for graphs embedded in arbitrary surfaces that subsumes the three variable version. We will work with this latter polynomial.

Definition 3.4 (Bollobás and Riordan [7]). Let G be a ribbon graph. Then

$$R(G; x, y, z, w) = \sum_{A \subseteq E(G)} (x-1)^{r(G)-r(A)} y^{n(A)} z^{k(A)-f(A)+n(A)} w^{t(A)} \in \mathbb{Z}[x, y, z, w]/\langle w^2 - w \rangle.$$

The (classical) Tutte polynomial $T(G; x, y)$ can be obtained from $R(G)$ as an evaluation:

$$(3.3) \quad T(G; x, y) = R(G; x, y-1, 1, 1).$$

In addition $T(G; x, y) = R(G; x, y-1, z, w)$ for all z and w if G is a plane graph. Much recent attention has been given to understanding how properties of the Tutte polynomial $T(G)$ extend to its topological analogue $R(G)$. Properties of the Tutte polynomial that extend to $R(G)$ include duality relations ([18, 29]), a spanning tree definition ([8, 34]), Brylawski's tensor product formula ([19]), the recipe theorem ([18]), and connections with knot theory ([10, 11, 12, 29]).

The polynomial $R(G)$ has been extended by the topochromatic polynomial, a multivariate generalization first defined in [29].

Definition 3.5 ([29]). Let G be an embedded graph. Let a, c , and w be indeterminates, and let $\mathbf{b} := \{b_e | e \in E(G)\}$ be a set of indeterminates indexed by $E(G)$. The *topochromatic* polynomial is

$$Z(G; a, \mathbf{b}, c, w) = \sum_{A \subseteq E(G)} a^{k(A)} \left(\prod_{e \in A} b_e \right) c^{f(A)} w^{t(A)} \in \mathbb{Z}[a, \mathbf{b}, c, w]/\langle w^2 - w \rangle.$$

The topochromatic polynomial generalization is in the spirit of the extensions of the Tutte polynomial to edge weighted graphs from [5], [32] and [35]. The introduction of the topochromatic polynomial was necessitated by some of the knot theoretical applications considered in [29]. We use the name ‘‘topochromatic polynomial’’ since $Z(G)$ is a multivariate extension of the dichromatic polynomial/random cluster model, $Z_T(G; u, v) = \sum u^{k(A)} v^{|A|}$, to embedded graphs.

$R(G)$ and the topochromatic polynomial are related as follows:

$$(3.4) \quad R(G; x, y, z, w) = (x-1)^{-k(G)} (yz)^{-v(G)} Z(G; (x-1)yz^2, \mathbf{b}, z^{-1}, w),$$

where the edge weights \mathbf{b} are given by setting $b_e = yz$ for each $e \in E(G)$. Also,

$$(3.5) \quad Z(G; a, \mathbf{b}, c, w) = (ac/b)^{k(G)} b^{v(G)} R(G; (ac+b)/b, bc, 1/c, w),$$

where the edge weights \mathbf{b} are given by setting $b_e = b$ for each $e \in E(G)$.

The multivariate signed Bollobás-Riordan polynomial, $Z_s(G)$, was introduced by Vignes-Tourneret in [33] as a multivariate version of Chmutov and Pak's signed Bollobás-Riordan polynomial, $R_s(G)$, from [10]. These are both polynomials of signed ribbon graphs (*i.e.* ribbon graphs with a sign $+$ or $-$ on each edge). The multivariate signed Bollobás-Riordan polynomial $Z_s(G)$, and hence $R_s(G)$, can be recovered from the topochromatic polynomial thus:

$$(3.6) \quad Z_s(G; q, \boldsymbol{\alpha}, c) = \left(\prod_{E_-(G)} q^{-1/2} \alpha_e \right) Z(G; q, \boldsymbol{\beta}, c, 1),$$

where $\boldsymbol{\beta} = \{\alpha_e | e \in E_+(G)\} \cup \{q\alpha_e^{-1} | e \in E_-(G)\}$, and E_{\pm} denotes the set of edges with sign \pm . Thus various results given here for the polynomials $Z(G)$ and $R(G)$ will also hold for $Z_s(G)$ and $R_s(G)$.

3.3. Las Vergnas' topological Tutte polynomial. Las Vergnas' topological Tutte polynomial $L(G; x, y, z)$, introduced in [23, 26] (see also [24]), is the first extension of the Tutte polynomial to embedded graphs that the authors are aware of, appearing in 1978. Thus far the Las Vergnas polynomial, $L(G)$, has not been as well studied as $R(G)$, although renewed interest in this polynomial may also be found in [3].

The Las Vergnas polynomial was first defined in terms of the combinatorial geometry of an embedded graph (*i.e.* its circuit matroid). The rank functions used in defining $L(G)$ are given in the context of $B(G)$ and $C(G)$, that is, the bond and circuit geometries of G . In Proposition 3.7 below, we will describe $L(G)$ in graph theoretical terms.

Definition 3.6 (Las Vergnas [23, 24, 26]). Let G be an embedded graph. Then

$$L(G; x, y, z) = \sum_{A \subseteq E(G)} (x-1)^{r(C(G)) - r_{C(G)}(A)} (y-1)^{|A| - r_{B(G^*)}(A)} z^{r(B(G^*)) - r(C(G)) - (r_{B(G^*)}(A) - r_{C(G)}(A))}.$$

By translating the notation and using Euler's formula, we can rewrite Las Vergnas' topological Tutte polynomial in a form that more clearly reveals how it encodes topological information.

Proposition 3.7.

$$(3.7) \quad L(G; x, y, z) = \sum_{A \subseteq E(G)} (x-1)^{r_G(A) - r_G(A)} (y-1)^{n_G(A) - (\gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c))/2} z^{(\gamma(G) - \gamma_G(A) + \gamma_{G^*}(A^c))/2},$$

where $A^c := E(G) - A$.

Proof. First note that $r(C(G)) = r(G)$ and $r_{C(G)}(A) = r_G(A)$. If M is a matroid and M^* its dual matroid, then $r_{M^*}(A) = |A| + r_M(M - A) - r(M)$. Then, since $B(G) = (C(G))^*$, we have $r(B(G^*)) = r(C(G^*)) = e(G^*) + r_{G^*}(\emptyset) - r(G^*) = n(G^*)$, and $r_{B(G^*)}(A) = r_{C(G^*)^*}(A) = |A| + r_{G^*}(A^c) - r(G^*)$, where we use the notation described in Subsection 2.2 and identify the edges of G and G^* .

By Euler's formula and the facts that $f_{G^*}(A^c) = f_G(A)$ and $f(G^*) = v(G)$, we have

$$\begin{aligned} 2r_{B(G^*)}(A) &= 2|A| - 2k_{G^*}(A^c) + 2k(G^*) \\ &= 2|A| - v_{G^*}(A^c) + |A^c| - f_{G^*}(A^c) - \gamma_{G^*}(A^c) + v(G^*) - e(G^*) + f(G^*) + \gamma(G^*) \\ &= |A| - v(G^*) + e(G^*) - f_{G^*}(A^c) - \gamma_{G^*}(A^c) + v(G^*) - e(G^*) + f(G^*) + \gamma(G^*) \\ &= |A| - f_G(A) - \gamma_{G^*}(A^c) + v(G) + \gamma(G^*) \\ &= v_G(A) - 2k_G(A) + \gamma_G(A) - \gamma_{G^*}(A^c) + v(G) + \gamma(G^*) \\ &= 2r_G(A) + \gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c). \end{aligned}$$

Also, using this computation for the exponent of z , we have:

$$\begin{aligned} r(B(G^*)) - r(C(G)) - (r_{B(G^*)}(A) - r_{C(G)}(A)) &= n(G^*) - r(G) - r_G(A) + r_G(A) - (\gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c))/2 \\ &= e(G^*) - v(G^*) + k(G^*) - v(G) + k(G) - (\gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c))/2 \\ &= e(G) - f(G) + 2k(G^*) - v(G) - (\gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c))/2 \\ &= \gamma(G) - (\gamma(G) + \gamma_G(A) - \gamma_{G^*}(A^c))/2 \\ &= (\gamma(G) - \gamma_G(A) + \gamma_{G^*}(A^c))/2. \end{aligned}$$

□

From the state sum given Proposition 3.7, one can immediately verify Las Vergnas' claim that $L(G)$ equals the Tutte polynomial when G is a plane graph, and hence $L(G)$ is a topological Tutte polynomial.

Furthermore, again from Proposition 3.7, we see that the Tutte polynomial of any graph may be recovered from $L(G)$ as follows.

Proposition 3.8 (Las Vergnas [26]). *For any embedded graph G ,*

$$(y-1)^{\gamma(G)} L(G; x, y, 1/(y-1)) = T(G; x, y).$$

4. THE TRANSITION POLYNOMIAL

In this section we describe the topological transition polynomial $Q(G)$, the action of the ribbon group upon it from [15], and its behaviour under an operation that doubles an edge of an embedded graph. The polynomials $P(G)$, $L(G)$, $R(G)$ and $Z(G)$ can all be recovered, in part or in full, from $Q(G)$, a fact that will be used in later sections to find identities for, and interpretations of evaluations of, $Z(G)$ and its specializations.

4.1. The topological transition polynomial. The generalized transition polynomial, $q(G; W, t)$, of [17] is a multivariate graph polynomial that generalizes Jaeger's transition polynomial of [21]. The transition polynomial assimilates the (plane) Penrose polynomial and agrees with the Tutte polynomial via a medial graph construction. In [15], the authors specialized the generalized transition polynomial to embedded graphs. This topological transition polynomial specializes to the topological Penrose polynomial and agrees with the topochromatic polynomial and with $R(G)$ on restricted sets of variables (and hence, to some extent, with $L(V)$ via $R(G)$). It is this link between $P(G)$, $Z(G)$ and $R(G)$ via the transition polynomial that we exploit in Subsection 5.2 to connect the polynomials. Although the generalized transition polynomial is more generally defined, here we restrict the transition polynomial to embedded medial graphs and particular weight systems determined by the embeddings. Because of these restrictions, we call the generalized transition polynomial specialized for this application the *topological transition polynomial*. Using the notation from Subsection 2.4, we recall the essential facts from [15] here.

Let F be a 4-regular graph having weight system W with values in a unitary ring \mathcal{R} . Then the state model formulation of the *generalized transition polynomial* is

$$q(F; W, t) = \sum_{s \in \text{St}(F)} \omega(s) t^{c(s)},$$

where the sum is over all graph states s of F . With this, and the medial weight system of Definition 2.2, we can define the topological transition polynomial.

Definition 4.1 ([15]). Let G be an embedded graph with embedded medial graph G_m . Then the *topological transition polynomial* of G is:

$$Q(G, (\alpha, \beta, \gamma), t) := q(G_m; W_m, t),$$

where W_m is the medial weight system of Definition 2.2.

The topological transition polynomial may be computed by repeatedly applying the following linear recursion relation at each $v \in V(G_m)$, and, when there are no more vertices of degree 4 to apply it to, evaluating each of the resulting closed curves to an independent variable t :

$$q(G_m, W_m, t) = \alpha_v q((G_m)_{wh(v)}, W_m, t) + \beta_v q((G_m)_{bl(v)}, W_m, t) + \gamma_v q((G_m)_{cr(v)}, W_m, t).$$

Pictorially, this relation is:

$$(4.1) \quad \begin{array}{c} \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \\ \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \end{array} = \alpha_v \begin{array}{c} \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \\ \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \end{array} + \beta_v \begin{array}{c} \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \\ \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \end{array} + \gamma_v \begin{array}{c} \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \\ \text{Diagram of a vertex } v \text{ with four incident edges forming a cross, shaded gray.} \end{array}.$$

4.2. Twisted duality and the transition polynomial. Twisted duality arose from an action of the symmetric group of order three, $\mathfrak{S} = \langle \delta, \tau \mid \delta^2, \tau^2, (\tau\delta)^3 \rangle$, on an edge of an embedded graph. The symmetric group also acts by permutation on the state weights at a vertex v , which are given by an ordered triple $(\alpha_v, \beta_v, \gamma_v)$, in a weight system W . Furthermore, if τ acts on $(\alpha_v, \beta_v, \gamma_v)$ as the permutation (1 3), and δ acts as the permutation (1 2), then it turns out that the action of \mathfrak{S} on weight systems is compatible with its action on embedded graphs in the sense described below. This compatibility was shown in [15] and provides us with a useful link between embedded graphs and graph polynomials. Just as we did for twisted duality, here we will describe the action on a weight system in a way that avoids a full discussion of the group actions, again the full details may be found in [15].

Let G_m be a canonically checkerboard coloured embedded medial graph of an embedded graph G with medial weight system W_m given by (α, β, γ) , and vertices indexed by the edges of G . If v_e is the vertex in G_m that corresponds to the edge e of G , we let

$$(\alpha, \beta, \gamma)^{\tau(e)} := \begin{cases} (\gamma_v, \beta_v, \alpha_v) & \text{if } v = v_e \\ (\alpha_v, \beta_v, \gamma_v) & \text{otherwise} \end{cases} \quad \text{and} \quad (\alpha, \beta, \gamma)^{\delta(e)} := \begin{cases} (\beta_v, \alpha_v, \gamma_v) & \text{if } v = v_e \\ (\alpha_v, \beta_v, \gamma_v) & \text{otherwise} \end{cases}.$$

For $\xi, \zeta \in \{\delta, \tau\}$, by setting $(\alpha, \beta, \gamma)^{\xi\zeta(e)} := ((\alpha, \beta, \gamma)^{\zeta(e)})^{\xi(e)}$, we obtain an action of $\mathfrak{S} = \langle \delta, \tau \mid \delta^2, \tau^2, (\tau\delta)^3 \rangle$ on a set of medial weight systems with a distinguished edge e .

If $A \subseteq E(G)$ and $\xi \in \mathfrak{S}$, we define $(\alpha, \beta, \gamma)^{\xi(A)}$ to be the medial weight system obtained by applying ξ to every edge in A . If ζ is also in \mathfrak{S} , it follows that $(\alpha, \beta, \gamma)^{\xi\zeta(A)} = (\alpha, \beta, \gamma)^{\zeta(A)\xi(A)}$ and that $(\alpha, \beta, \gamma)^{\xi(A)\zeta(B)} = ((\alpha, \beta, \gamma)^{\xi(A)})^{\zeta(B)}$, for $B \subseteq E(G)$.

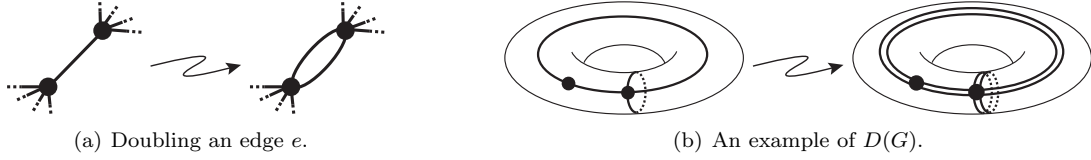


FIGURE 3. Doubling edges of an embedded graph.

We can now state the twisted duality relation for the topological transition polynomial. This says that the topological transition polynomial of the medial graph of G is the same as that of the medial graph of any of the twisted duals, provided the weight system is appropriately permuted. We will apply this twisted duality relation in the subsequent sections to derive new duality properties for the Penrose and topological Tutte polynomials via the transition polynomial.

Theorem 4.2 ([15]). *Let G be an embedded graph with embedded medial graph G_m , and let $\Gamma = \prod_{i=1}^k \xi_i(A_i)$, where $A_i \subseteq E(G)$, and $\xi_i \in \mathfrak{G}$, for each i . Then,*

$$Q(G; (\alpha, \beta, \gamma), t) = Q(G^\Gamma, (\alpha, \beta, \gamma)^\Gamma, t).$$

4.3. Doubling edges and the transition polynomial. In this subsection we introduce an operation on an embedded graph that acts by replacing an edge by a pair of parallel edges. We will see that embedded graphs that are related by this operation have closely related transition polynomials. We use this fact to provide an alternative connection between $P(G)$ and $R(G)$. With this connection, we are able to determine new combinatorial interpretations of evaluations of $R(G)$.

Definition 4.3. Let e be an edge of an embedded graph G . We let $D_e(G)$ denote the embedded graph obtained from G by embedding a new edge parallel to e so that the two edges bound a face of degree 2, as in Figure 3(a). We say that $D_e(G)$ is obtained from G by *doubling the edge e* . Furthermore, if $A \subseteq E(G)$ we let $D_A(G)$ denote the graph obtained by doubling all of the edges in A , and we denote $D_{E(G)}(G)$ by $D(G)$.

An example of $D(G)$, for a graph G embedded in the torus, is given in Figure 3(b).

We record the following basic observations about $D(G)$ for use later.

Proposition 4.4. *Let G be an embedded graph. Then*

- (1) $D(G)$ is checkerboard colourable,
- (2) $D(G)$ is orientable if and only if G is,
- (3) $\gamma(D(G)) = \gamma(G)$.

The significance of $D_A(G)$ here is that the transition polynomials of G and $D_A(G)$ are related in a simple way, and this relation allows us to express $R(G)$ in terms of $P(D_A(G))$. To relate the transition polynomials of $D_A(G)$ and G , we introduce the following notation for medial weight systems.

Definition 4.5. Let G be an embedded graph and $A \subseteq E(G)$. We use the notation that if e is an edge in G with $e \in A$, then e and e' denote the edge and its double in $D_A(G)$. Furthermore, if v is the vertex in G_m corresponding to e , then v and v' are the vertices in $(D_A(G))_m$ corresponding to e and e' .

With this notation, and with W_m as the medial weight system for $D_A(G)$ that assigns a state weight $(\alpha_w, \beta_w, \gamma_w)$ to a vertex w , we let $D_A(W_m)$ denote the weight system for G that assigns a state weight

- $(\alpha_v, \beta_v, \gamma_v)$ to a vertex v that does not correspond to an edge in A ,
- $(\alpha_v \alpha_{v'} t + \alpha_v (\beta_{v'} + \gamma_{v'}) + \alpha_{v'} (\beta_v + \gamma_v), \beta_v \beta_{v'} + \gamma_v \gamma_{v'}, \beta_v \gamma_{v'} + \beta_{v'} \gamma_v)$ to a vertex v that corresponds to an edge in A .

The following theorem says that the actions of D_A on a graph and on a weight system are compatible.

Theorem 4.6. *Let G be an embedded graph, $A \subseteq E(G)$, and W_m be a medial weight system for $D_A(G)_m$. Then*

$$q((D_A(G))_m; W_m, t) = q(G_m; D_A(W_m), t).$$

Proof. It is enough to prove the theorem when A only contains a single edge e . Applying the linear recursion from Equation (4.1) to the vertex v in G corresponding to e , and using the weight system $D_A(W_m)$ gives

$$(4.2) \quad \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = (\alpha_v \alpha_{v'} t + \alpha_v (\beta_{v'} + \gamma_{v'}) + \alpha_{v'} (\beta_v + \gamma_v)) \begin{array}{c} \diagup \\ \diagdown \end{array} \\ + (\beta_v \beta_{v'} + \gamma_v \gamma_{v'}) \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} + (\beta_v \gamma_{v'} + \beta_{v'} \gamma_v) \begin{array}{c} \diagdown \\ \bullet \\ \diagup \end{array}.$$

The medial graph $D_e(G)_m$ can be obtained from G_m by the following replacement:



leaving the rest of the embedded graph unchanged. Applying the linear recursion from Equation (4.1), using the weight system W_m , to the vertices v and v' in $D_e(G)_m$ gives:

$$\begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = \alpha_v \alpha_{v'} \begin{array}{c} \diagup \\ \diagdown \end{array} + \alpha_v \beta_{v'} \begin{array}{c} \diagup \\ \diagdown \end{array} + \beta_v \alpha_{v'} \begin{array}{c} \diagdown \\ \diagup \end{array} + \alpha_v \gamma_{v'} \begin{array}{c} \diagup \\ \diagdown \end{array} + \gamma_v \alpha_{v'} \begin{array}{c} \diagdown \\ \diagup \end{array} \\ + \beta_v \beta_{v'} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} + \gamma_v \beta_{v'} \begin{array}{c} \diagup \\ \diagdown \end{array} \begin{array}{c} \diagdown \\ \diagup \end{array} + \beta_v \gamma_{v'} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array} + \gamma_v \gamma_{v'} \begin{array}{c} \diagdown \\ \diagup \end{array} \begin{array}{c} \diagup \\ \diagdown \end{array},$$

where we have omitted the checkerboard colouring on the right-hand side for clarity. After evaluating the closed curve in the $\alpha_v \alpha_{v'}$ term to t , deforming the curves, and collecting terms, we see that this is exactly the expression on the left-hand side of Equation (4.2). \square

5. RELATIONS AMONG POLYNOMIALS

In this section we present new connections among the topological graph polynomials $P(G)$, $R(G)$, $Z(G)$, $L(G)$, and $Q(G)$. These connections among the polynomials will lead to several new combinatorial identities and interpretations of evaluations of these polynomials.

5.1. Recovering polynomials from the transition polynomial. Our interest in the transition polynomial here lies in the fact that it assimilates a number of graph polynomials. We will now describe how the polynomials $P(G)$, $R(G)$ and $Z(G)$ are related to the topological transition polynomial.

For the relation with the Penrose polynomial, we let G be an embedded graph and G_m be its embedded medial graph equipped with the canonical checkerboard colouring. We define the weight system $W_P(G_m)$ by

$$W_P(G_m) : \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = 1 \begin{array}{c} \diagup \\ \diagdown \end{array} - 1 \begin{array}{c} \diagdown \\ \diagup \end{array}.$$

The following proposition relates the Penrose polynomial to the transition polynomial using the weight system W_P .

Proposition 5.1 ([16]). *Let G be an embedded graph and G_m be its canonically checkerboard coloured medial graph. Then*

$$P(G; \lambda) = q(G_m; W_P, \lambda) = Q(G; (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda).$$

For the relation with the topochromatic polynomial, we let G be an embedded graph and G_m be its embedded medial graph equipped with the canonical checkerboard colouring. We define the weight system $W_Z(G_m)$ by

$$W_Z(G_m) : \begin{array}{c} \diagup \\ \bullet \\ \diagdown \end{array} = b_v \begin{array}{c} \diagup \\ \diagdown \end{array} + 1 \begin{array}{c} \diagdown \\ \diagup \end{array}.$$

With this weight system we obtain a relation between $Z(G)$ and $Q(G)$.

Proposition 5.2. *If G is an embedded graph, and $(\mathbf{b}, \mathbf{1}, \mathbf{0})$ corresponds to the weight system W_Z , then*

$$Z(G; \mathbf{1}, \mathbf{b}, c, 1) = q(G_m; W_z, c) = Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0}), c).$$

Proof. By definition,

$$Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0}), c) = \sum_{s \in \mathcal{Z}(G_m)} \omega_Z(s) c^{c(s)} = \sum_{s \in \mathcal{Z}(G_m)} \left(\prod_{v_e \in Wh(s)} b_e \right) c^{c(s)},$$

where the sum is over all graph states s with no crossing states, $c(s)$ is the number of components in the state s , and where $Wh(s)$ is the set of vertices with white split states in the graph state s .

We can define a bijection between the set of embedded spanning subgraphs of G and the set of graph states of G_m by associating an edge set A_s of G by setting $e \in A_s$ if and only if the vertex state $v_e \in Wh(s)$. It is then clear that, for every graph state, $c(s) = f(A_s)$. By using this bijection, we have

$$Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0}), c) = \sum_{s \in \mathcal{Z}(G_m)} \left(\prod_{v_e \in Wh(s)} b_e \right) c^{c(s)} = \sum_{A \subseteq E(G)} \left(\prod_{e \in A} b_e \right) c^{f(H)} = Z(G; \mathbf{1}, \mathbf{b}, c, 1).$$

□

5.2. Twisted duals and the Penrose and topological Tutte polynomials. The relation between the topochromatic polynomial and the transition polynomial now allows us to express the topological Penrose polynomial as an evaluation of $Z(G)$, and of $R(G)$. We will use this connection between the two polynomials to find some new combinatorial interpretations of evaluations of topological Tutte polynomials. We will also reformulate the Four Colour Theorem in terms of $R(G)$.

Theorem 5.3. *Let G be an embedded graph. Then*

$$P(G^*; \lambda) = Z(G^\times; \mathbf{1}, -\mathbf{1}, \lambda, 1).$$

Proof. We have

$$P(G; \lambda) = Q(G; (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda) = Q(G^{\tau^\delta(E(G))}; (-\mathbf{1}, \mathbf{1}, \mathbf{0}), \lambda) = Z(G^{\tau^\delta(E(G))}; \mathbf{1}, -\mathbf{1}, \lambda, 1),$$

where the first equality follows from Proposition 5.1, the second follows from Theorem 4.2, and the third equality from Proposition 5.2. The result then follows by applying this to $G^* = G^{\delta(E(G))}$. □

Combining Theorem 5.3 with Equation (3.4), we can now express the Penrose polynomial in terms of the topological Tutte polynomial.

Corollary 5.4. *Let G be an embedded graph. Then*

$$(-1)^{r(G^\times)} \lambda^{k(G^\times)} R(G^\times; 1 - \lambda, -\lambda, 1/\lambda, 1) = P(G^*; \lambda).$$

5.3. Doubling edges and combinatorial interpretations of the topochromatic polynomial. The following theorem is a consequence of Theorem 4.6. It states that the Penrose polynomial of $D(G)$ arises as an evaluation of the topochromatic polynomial $Z(G)$. We will use this to obtain new combinatorial descriptions of evaluations of $Z(G)$ and $R(G)$ and of the Tutte polynomial $T(G)$.

Theorem 5.5. *Let G be an embedded graph. Then*

$$Z(G; \mathbf{1}, \boldsymbol{\lambda} - \mathbf{2}, \lambda, 1) = P(D(G); \lambda).$$

Proof. We have

$$P(D(G); \lambda) = Q(D(G); (\mathbf{1}, \mathbf{0}, -\mathbf{1}), \lambda) = Q(G; (\boldsymbol{\lambda} - \mathbf{2}, \mathbf{1}, \mathbf{0}), \lambda) = Z(G; \mathbf{1}, \boldsymbol{\lambda} - \mathbf{2}, \lambda, 1),$$

where the first equality follows from Proposition 5.1, the second from Theorem 4.6, and the third from Proposition 5.2. □

Equation (3.4) then gives the following corollary which relates $R(G)$ and the Penrose polynomial.

Corollary 5.6. *Let G be an embedded graph. Then*

$$R\left(G; \frac{2(\lambda-1)}{\lambda-2}, \lambda(\lambda-2), \frac{1}{\lambda}, 1\right) = \left(\frac{\lambda-2}{\lambda}\right)^{k(D(G))} \left(\frac{1}{\lambda-2}\right)^{v(D(G))} P(D(G); \lambda).$$

Since $T(G)$ and $R(G)$ agree on plane graphs via Equation 3.3, the Tutte polynomial of a plane graph is also related to the Penrose polynomial of its doubled graph.

Corollary 5.7. *If G is a plane graph. Then*

$$T\left(G; \frac{2(\lambda-1)}{\lambda-2}, (\lambda-1)^2\right) = \left(\frac{\lambda-2}{\lambda}\right)^{k(D(G))} \left(\frac{1}{\lambda-2}\right)^{v(D(G))} P(D(G); \lambda).$$

5.4. Relating Las Vergnas' and Bollobás and Riordan's topological Tutte polynomials. Since both $L(G)$ and $R(G)$ essentially agree with the Tutte polynomial on plane graphs, they also agree with each other on plane graphs. However, here we see in Proposition 5.8 that $L(G)$ and $R(G)$ are in fact very closely related for all embedded graphs. In Proposition 5.10, we show that an evaluation of $R(G)$ is the generating function of non-crossing graph states. This allows us to show that 1-variable specializations of Bollobás and Riordan's, and Las Vergnas' topological Tutte polynomials agree. Moreover, by results of [3], it follows that 1-variable specializations of $R(G)$, and Kruskal's topological Tutte polynomial from [22] will also agree. Due to Proposition 5.8, we expect there are many other ways of relating $L(G)$ and $R(G)$.

We begin by collecting together the topological contributions in the expression for $L(G)$ given in Proposition 3.7 to obtain a particularly simple form of the Las Vergnas polynomial $L(G)$.

Proposition 5.8. *Let G be an embedded graph, then*

$$(z(y-1))^{\gamma(G)} L\left(G; x, y, \frac{1}{z^2(y-1)}\right) = \sum_{A \subseteq E(G)} (x-1)^{r_G(G)-r_G(A)} (y-1)^{n_G(A)} z^{\gamma_G(A)-\gamma_{G^*}(A^c)}.$$

Note that, using Euler's formula, we can rewrite the Bollobás-Riordan's polynomial $R(G)$ in the form

$$R(G; x, y, z, 1) = \sum_{A \subseteq E(G)} (x-1)^{r_G(G)-r_G(A)} (y-1)^{n_G(A)} z^{\gamma_G(A)}.$$

Comparing this state sum for $R(G)$ with the state sum for $L(G)$ given Proposition 5.8 illuminates the relation between these two topological Tutte polynomials.

Since $L(G)$ and $R(G)$ both specialize to the Tutte polynomial, we obtain a further relation between these two polynomials:

Proposition 5.9. *Let G be an embedded graph. Then*

$$R(G; x, y, 1, 1) = y^{\gamma(G)} L(G; x, y+1, 1/y).$$

Proof. By Equation 3.3 and Proposition 3.8,

$$R(G; x, y-1, 1, 1) = T(G; x, y) = (y-1)^{\gamma(G)} L(G; x, y, 1/(y-1)).$$

□

We now show what a one-variable evaluation of $R(G)$ counts, and use this to give another relation between $R(G)$ and $L(G)$.

Proposition 5.10. *Let G be a connected embedded graph and let $f_k(G_m)$ be the number of k -component graph states of its medial graph G_m without crossings. Then*

$$tR(G; t+1, t, 1/t, 1) = \sum_{k \geq 1} f_k(G_m) t^k.$$

Proof.

$$\sum_{k \geq 1} f_k(G_m) t^k = Q(G; (\mathbf{1}, \mathbf{1}, \mathbf{0}), t) = Z(G; 1, \mathbf{1}, t, 1) = tR(G; t+1, t, \frac{1}{t}, 1).$$

Here the first equality comes from the definition of the transition polynomial, the second from Proposition 5.2, and the final equality by Equation (3.4). □

Theorem 5.11. *If G is a graph embedded on the plane or projective plane. Then*

$$L(G; t+1, t+1, 1) = R(G; t+1, t, 1/t, 1);$$

and if G is embedded in the torus, then

$$L_2(G; t+1, t+1, 1) + tL_1(G; t+1, t+1, 1) + L_0(G; t+1, t+1, 1) = R(G; t+1, t, 1/t, 1),$$

where, if we view $L(G; x, y, z)$ as a polynomial in $(\mathbb{Z}[x, y])[z]$, then L_i is the coefficient of z^i in $L(G; x, y, z)$.

Proof. Let F be the canonically checkerboard coloured medial graph of G (so that $F_{bl} = G$). Las Vergnas proved in Proposition 4.1 of [24] that $tL(F_{bl}; t+1, t+1, 1) = \sum_{k \geq 1} f_k(F)t^k$ when F is on the sphere or the projective plane; and that $L_2(F_{bl}; t+1, t+1, 1) + tL_1(F_{bl}; t+1, t+1, 1) + L_0(F_{bl}; t+1, t+1, 1) = \sum_{k \geq 1} f_k(F)t^{k-1}$, when F is on the torus. The results then follow by Proposition 5.10. \square

6. IDENTITIES AND EVALUATIONS OF TOPOLOGICAL GRAPH POLYNOMIALS

We can use the preceding results to obtain identities and both topological and combinatorial interpretations of some evaluations of the topochromatic polynomial. We provide translations of these results to $R(G)$. These results may also be translated, at least in part, to $L(G)$ using the relationships in Section 5, but we leave this to the reader.

6.1. Twisted duals and interpretations of $Z(G)$ and $R(G)$. We begin by using the relationship between $P(G)$ and $Z(G)$ given in Theorem 5.3, and the identities and evaluations of Subsection 3.1, to find identities for, and interpretations of, valuations of $Z(G)$.

Theorem 6.1. *Let G be an embedded graph.*

(1) *If $H := (G^\times)^*$ is plane, connected and cubic, then the number of edge 3-colourings of H is*

$$Z(G; 1, -\mathbf{1}, 3, 1) = (-1/4)^{v(H)/2} Z(G; 1, -\mathbf{1}, -2, 1).$$

(2) *If G^\times is plane, then*

$$Z(G; 1, -\mathbf{1}, \lambda, 1) = \sum_{A \subseteq E(H)} \chi((G^{\delta(A)})^\times; \lambda).$$

(3) *If $(G^\times)^*$ is orientable and checkerboard colourable, then*

$$Z(G; 1, -\mathbf{1}, 2, 1) = 2^{f(G^\times)}.$$

(4) *If $(G^\times)^*$ is orientable and checkerboard colourable, then*

$$Z(G; 1, -\mathbf{1}, -1, 1) = (-1)^{v(G)} 2^{e(G)}.$$

(5) *Let $H := (G^\times)^*$, then for each $k \in \mathbb{N}$,*

$$Z(G; 1, -\mathbf{1}, k, 1) = \sum_{\sigma \in \text{Adm}(H, k)} (-1)^{\text{cr}(\sigma)},$$

where $\text{cr}(\sigma)$ denotes the number of crossing states in the admissible k -valuation σ of H_m .

(6) *Let $H := (G^\times)^*$ be orientable and checkerboard colourable, then for each $k \in \mathbb{N}$,*

$$Z(G; 1, -\mathbf{1}, -k, 1) = (-1)^{f(H)} \sum_{\sigma \in \text{Per}(H, k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

Proof. Item 1 follows from Theorem 5.3 and Item 1 of Theorem 3.2.

For Item 2, observe that Theorem 5.3 and Item 2 of Theorem 3.2 give

$$Z(G; 1, -\mathbf{1}, \lambda, 1) = \sum_{A \subseteq E((G^\times)^*)} \chi(((G^\times)^*)^{\tau(A)}; \lambda).$$

Rewriting the exponent, with $A^c := E(G) - A$, gives

$$\delta\tau(E(G))\tau(A)\delta(E(G)) = \delta\tau\delta\tau(A)\delta\delta\tau(A^c) = \tau\delta(A)\tau(A^c) = (G^{\delta(A)})^\times,$$

and the result follows.

Item 3 follows from Theorem 5.3 and Item 3 of Theorem 3.2.

Item 4 follows from Theorem 5.3 and Item 4 of Theorem 3.2, upon noting that $f((G^\times)^*) = v(G^\times) = v(G)$.

Item 5 follows from Theorem 5.3 and Theorem 3.3.

Item 6 follows from Theorem 5.3 and Theorem 3.3. \square

Once we know what to look for, Item 4 of Theorem 6.1 can easily be shown to hold for all oriented graphs and those in the projective plane.

Proposition 6.2. *If G is graph embedded in an orientable surface or a projective plane, then*

$$Z(G; 1, -\mathbf{1}, -1, -1) = (-1)^{v(G)} 2^{e(G)}.$$

Proof. Using Definition 3.5,

$$Z(G; 1, -\mathbf{1}, -1, 1) = \sum_{A \subseteq E(G)} (-1)^{|A|+f(A)+t(A)} = \sum_{A \subseteq E(G)} (-1)^{v(A)+\gamma(A)+t(A)} = (-1)^{v(G)} 2^{e(G)},$$

where the second equality holds by Euler's formula, and the third since $\gamma(A) \equiv t(A) \pmod{2}$ on an orientable surface or the real projective plane. \square

The following corollary reinterprets the results of Theorem 6.1 and Proposition 6.2 in terms of $R(G)$, using Equation (3.5).

Corollary 6.3. *Let G be an embedded graph.*

(1) *If $H := (G^\times)^*$ is plane and is connected and cubic, then the number of edge 3-colourings of H is*

$$(-1)^{r(G)} 3 R(G; -2, -3, 1/3, 1) = (-1)^{v(H)/2+v(G)} (1/4)^{v(H)/2} 2 R(G; 3, 2, -1/2, 1).$$

(2) *If G^\times is plane, then*

$$R(G; 1 - \lambda, -\lambda, 1/\lambda, 1) = (-1)^{r(G)} (1/\lambda)^{k(G)} \sum_{A \subseteq E(H)} \chi((G^{\delta(A)})^\times; \lambda).$$

(3) *If $(G^\times)^*$ is orientable and checkerboard colourable, then*

$$(-1)^{r(G)} 2^{k(G)} R(G; -1, -2, 1/2, 1) = 2^{f(G^\times)}.$$

(4) *If $H := (G^\times)^*$ is orientable or is embedded in the projective plane, then*

$$R(G; 2, 1, -1, 1) = (-1)^{k(G)} 2^{e(G)}.$$

(5) *Let $H := (G^\times)^*$, then for each $k \in \mathbb{N}$,*

$$R(G; 1 - k, k, 1/k, 1) = (-1)^{r(G)} (1/k)^{k(G)} \sum_{\sigma \in \text{Adm}(H, k)} (-1)^{\text{cr}(\sigma)},$$

where $\text{cr}(\sigma)$ denotes the number of crossing states in the admissible k -valuations σ of H_m .

(6) *Let $H := (G^\times)^*$ be orientable and checkerboard colourable, then for each $k \in \mathbb{N}$,*

$$R(G; 1 + k, -k, -1/k, 1) = (-1)^{v(G)+f(H)} (1/k)^{k(G)} \sum_{\sigma \in \text{Per}(H, k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

We can also use the relation between the Penrose and topological Tutte polynomials of Corollary 5.4 to obtain a new formulation of the Four Colour Theorem in terms of $R(G)$.

Corollary 6.4. *The following statements are equivalent:*

(1) *the Four Colour Theorem is true;*

(2) *for every connected, loopless plane graph G , $(-1)^{f(G)} R(G^\times; -2, -3, 1/3, 1) < 0$;*

(3) *for every connected, loopless plane graph G , $(-1)^{f(G)} R(G^\times; -3, -4, 1/4, 1) < 0$.*

Proof. Corollary 9 of [1] states that the Four Colour Theorem is equivalent to showing that $P(G; 3) > 0$ for all connected, bridgeless plane graphs G . By Corollary 5.4, this is equivalent to showing that $(-1)^{r(G)} 3^{k(G)} R((G^*)^\times; -2, -3, 1/3, 1) > 0$ for all connected, bridgeless plane graphs G ; which is true if and only if $(-1)^{v(G)} R((G^*)^\times; -2, -3, 1/3, 1) < 0$ for all connected, bridgeless plane graphs G . This is equivalent to showing that $(-1)^{f(G)} R(G^\times; -2, -3, 1/3, 1) < 0$ for all connected, loopless plane graphs G .

The remaining equivalence is shown in a similar way but using the fact from [1] that the Four Colour Theorem is equivalent to showing that $P(G; 4) > 0$ for all connected, bridgeless plane graphs G \square

6.2. Doubling edges and interpretations of the topochromatic polynomial. By combining Theorem 5.5 with results on the Penrose polynomial from Subsection 3.1, we are able to find additional combinatorial interpretations for valuations of the topochromatic polynomial, for $R(G)$, and for the Tutte polynomial.

Theorem 6.5. *Let G be an embedded graph.*

(1) *If G is plane, then*

$$Z(G; 1, \boldsymbol{\lambda} - \mathbf{2}, \lambda, 1) = \sum_{A \subseteq E(D(G))} \chi((D(G)^{\tau(A)})^*; \lambda).$$

(2) *For each $k \in \mathbb{N}$,*

$$Z(G; 1, \mathbf{k} - \mathbf{2}, k, 1) = \sum_{\sigma \in \text{Adm}(D(G), k)} (-1)^{\text{cr}(\sigma)},$$

where $\text{cr}(\sigma)$ denotes the number of crossing states in the admissible k -valuations σ of $(D(G))_m$. (If G is plane this equals the number of admissible k -valuations of $(D(G))_m$.)

(3) *Let G be orientable, then for each $k \in \mathbb{N}$,*

$$Z(G; 1, -\mathbf{k} - \mathbf{2}, -k, 1) = (-1)^{v(G)} \sum_{\sigma \in \text{Per}(D(G), k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

Proof. Item 1 follows from Theorem 5.5 and Item 2 of Theorem 3.2. Item 2 follows from Theorem 5.5 and Theorem 3.3. Item 3 follows from Theorem 5.5 and Theorem 3.3. \square

Again, equation (3.5) allows us to rephrase the preceding results in terms of $R(G)$.

Corollary 6.6. *Let G be an embedded graph.*

(1) *If G is plane, then*

$$R\left(G; \frac{2(\lambda - 1)}{\lambda - 2}, \lambda(\lambda - 2), \frac{1}{\lambda}, 1\right) = \left(\frac{\lambda - 2}{\lambda}\right)^{k(G)} \left(\frac{1}{\lambda - 2}\right)^{v(G)} \sum_{A \subseteq E(D(G))} \chi((D(G)^{\tau(A)})^*; \lambda).$$

(2) *For each $k \in \mathbb{N}$,*

$$R\left(G; \frac{2(k - 1)}{k - 2}, k(k - 2), \frac{1}{k}, 1\right) = \left(\frac{k - 2}{k}\right)^{k(G)} \left(\frac{1}{k - 2}\right)^{v(G)} \sum_{\sigma \in \text{Adm}(D(G), k)} (-1)^{\text{cr}(\sigma)},$$

where $\text{cr}(\sigma)$ denotes the number of crossing states in the admissible k -valuation σ of $D(G)_m$.

(3) *Let G be orientable and checkerboard colourable, then for each $k \in \mathbb{N}$,*

$$R\left(G; \frac{2(k + 1)}{k + 2}, k(k + 2), \frac{-1}{k}, 1\right) = \left(\frac{k + 2}{k}\right)^{k(G)} \left(\frac{1}{k + 2}\right)^{v(G)} \sum_{\sigma \in \text{Per}(D(G), k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

Since $R(G)$ and $T(G)$ agree on plane graphs (see Equation 3.3), the preceding results specialize to the classical Tutte polynomial.

Corollary 6.7. *Let G be a plane graph.*

(1) We have

$$T\left(G; \frac{2(\lambda-1)}{\lambda-2}, \lambda^2 - 2\lambda + 1\right) = \left(\frac{\lambda-2}{\lambda}\right)^{k(G)} \left(\frac{1}{\lambda-2}\right)^{v(G)} \sum_{A \subseteq E(D(G))} \chi((D(G)^{\tau(A)})^*; \lambda).$$

(2) For each $k \in \mathbb{N}$,

$$T\left(G; \frac{2(k-1)}{k-2}, k^2 - 2k + 1\right) = \left(\frac{k-2}{k}\right)^{k(G)} \left(\frac{1}{k-2}\right)^{v(G)} |\text{Adm}(D(G), k)|.$$

(3) For each $k \in \mathbb{N}$,

$$T\left(G; \frac{2(k+1)}{k+2}, k^2 + 2k + 1\right) = \left(\frac{k+2}{k}\right)^{k(G)} \left(\frac{1}{k+2}\right)^{v(G)} \sum_{\sigma \in \text{Per}(D(G), k)} 2^{m(\sigma)},$$

where $m(\sigma)$ is the number of total vertices in σ .

It follows from Theorem 5.5 and Item 3 of Theorem 3.2 that when G is orientable, $Z(G; 1, \mathbf{0}, 2, 1) = 2^{v(G)}$. It is straight-forward to show that this identity follows in more generality.

Proposition 6.8.

(1) If G is an embedded graph, then

$$Z(G; a, \mathbf{0}, c, w) = (ac)^{v(G)}.$$

(2) If G is an embedded graph, then, with the convention that we simplify the left-hand side before evaluating,

$$\left[\left(\frac{ac}{b}\right)^{k(G)} b^{v(G)} R\left(G; \frac{ac+b}{b}, bc+1, \frac{1}{c}, w\right) \right]_{b=0} = (ac)^{v(G)}.$$

(3) For any graph G , with the convention that we simplify the left-hand side before evaluating,

$$\left[\left(\frac{ac}{b}\right)^{k(G)} b^{v(G)} T\left(G; \frac{ac+b}{b}, bc+1\right) \right]_{b=0} = (ac)^{v(G)}.$$

Proof. For Item 1,

$$Z(G; a, \mathbf{0}, c, w) = \sum_{A \subseteq E(G)} a^{k(A)} \left(\prod_{e \in A} b_e\right) c^{f(A)} w^{t(A)} = a^{k(\emptyset)} c^{f(\emptyset)} = (ac)^{v(G)}.$$

Item 2 follows from Item 1 by Equation 3.5.

For Item 3,

$$\begin{aligned} \left(\frac{ac}{b}\right)^{k(G)} b^{v(G)} T\left(G; \frac{ac+b}{b}, bc+1\right) &= \left(\frac{ac}{b}\right)^{k(G)} b^{v(G)} \sum_{A \subseteq E(G)} \left(\frac{ac}{b}\right)^{r(G)-r(A)} (bc)^{n(A)} \\ &= \sum_{A \subseteq E(G)} a^{k(A)} b^{|A|} c^{|A|-v(A)+2k(A)}. \end{aligned}$$


Evaluating the final expression at $b = 0$ gives $a^{k(\emptyset)} c^{-v(\emptyset)+2k(\emptyset)}$ which equals $(ac)^{v(G)}$, as required. \square

We note that it also follows from Theorem 5.5 and Theorem 3.2 that $Z(G; 1, -\mathbf{3}, -1, 1) = (-1)^{v(G)} 4^{e(G)}$. This identity can also be readily obtained directly from the definition of Z .

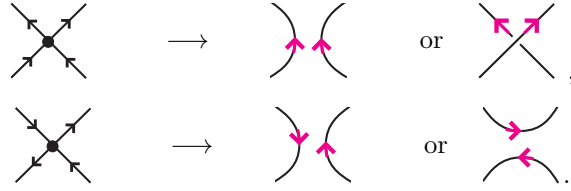
6.3. Circuits and topological Tutte polynomials. The classical Tutte polynomial gives a generating function for the components in non-crossing graph states of 4-regular checkerboard coloured plane graphs (see Martin [28], Las Vergnas [25, 27], and Ellis-Monaghan [13, 14]). In [24], Las Vergnas gave analogous results for his topological Tutte polynomial for 4-regular checkerboard coloured graphs embedded in the torus and projective plane.

We will extend Proposition 5.10, which states $tR(G; t+1, t, 1/t, 1) = \sum_{k \geq 1} f_k(G) t^k$, so that it includes circuits with crossing states. We need the following lemma.

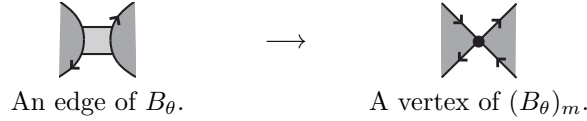
Lemma 6.9. *Let F be any 4-regular abstract graph and let θ be any Eulerian orientation of F (i.e. direct the edges of F so that at each vertex there are two incoming and two outgoing edges). Then there is a checkerboard coloured embedding of F such that at each vertex the crossing state corresponds to the pairing together of the two incoming edges and the two outgoing edges (the anticoherent pairing), and the white and black splits correspond to an incoming edge paired with an outgoing edge (the coherent pairings).*

Proof. To prove the lemma, we need to construct a checkerboard colourable embedding of F such that every vertex is of the form . We will do this by first taking an arbitrary embedding \tilde{F} of F , then forming a particular cycle family graph. The required embedding of F then arises as the medial graph of this cycle family graph.

Let \tilde{F} be any embedding of F , with Eulerian orientation inherited from θ . At each vertex of \tilde{F} , choose an arrow marked state corresponding to a coherent pairing with the arrows following the orientation given by θ , as follows:



This gives a (non-unique) cycle family graph B_θ . It is clear, by considering the following diagram, that $(B_\theta)_m$, which is an embedding of F by Theorem 2.4, has the desired property.



□

The language of embedded graphs and cycle family graphs allows us to extend Proposition 5.10 and express the generating function for *all* graph states as an evaluation of $R(G)$.

Theorem 6.10. *Let F be any 4-regular abstract graph, and let $\tilde{f}_k(F)$ be the number of k -component graph states of F . Then*

$$\sum_{k \geq 1} \tilde{f}_k(F) (2t)^k = \sum_{\theta} R(B_\theta; t+1, t, t, 1),$$

where the right-hand sum is over all Eulerian orientations θ of F , and B_θ is the blackface graph of an embedding with $(B_\theta)_m \cong F$ given by Lemma 6.9.

Proof. Let W_1 be the weight system that assigns a weight of 1 to every vertex state; let W_θ be the weight system that assigns a 1 to the two coherent states at each vertex and a 0 to the anticoherent states; and let W_m be the medial weight system $(\mathbf{1}, \mathbf{1}, \mathbf{0})$. First note that $\sum_{k \geq 1} \tilde{f}_k(F) t^k = q(F; W_1, t)$. Then a result of Las Vergnas [27], given in terms of the Martin polynomials and translated here to the transition polynomial, says that

$$t q(F; W_1, 2t) = \sum_{\theta} q(F; W_\theta, t),$$

where the sum is over all Eulerian orientations θ of F . By Lemma 6.9, for all θ there is an embedding of F , namely $(B_\theta)_m$, such that the white and black splits correspond to the coherent pairings in θ . Thus,

$$q(F; W_\theta, t) = q((B_\theta)_m; W_m, t) = Q(B_\theta; (\mathbf{1}, \mathbf{1}, \mathbf{0}), t).$$

From the proof of Proposition 5.10, $Q(B_\theta; (\mathbf{1}, \mathbf{1}, \mathbf{0}), t) = tR(B_\theta; t+1, t, 1/t, 1)$. Therefore,

$$t \sum_{k \geq 1} \tilde{f}_k(F) (2t)^k = \sum_{\theta} tR(B_\theta; t+1, t, 1/t, 1),$$

from which the result follows. □

6.4. Partial duality and the topochromatic polynomial. In this section we show how the ribbon group action on the transition polynomial provides a framework for understanding the recent duality results for the $Z(G)$ and for $R(G)$ that have been studied by various authors (see [6, 9, 18, 29, 33]).

Let G be an embedded graph and let G_m be its embedded medial graph equipped with the canonical checkerboard colouring. Then, for $A \subseteq E(G)$, the weight system $W_Z^{\delta(A)}(G_m)$ is given by reversing the roles of b_e and 1 whenever e is in A , thus:

$$W_Z^{\delta(A)}(G_m) : \begin{cases} \text{if } e \notin A \text{ then} & \begin{array}{c} \text{X} \\ \text{v}_e \end{array} = b_e \begin{array}{c} \text{C} \\ \text{C} \end{array} + \begin{array}{c} \text{D} \\ \text{D} \end{array} \\ \text{if } e \in A \text{ then} & \begin{array}{c} \text{X} \\ \text{v}_e \end{array} = \begin{array}{c} \text{C} \\ \text{C} \end{array} + b_e \begin{array}{c} \text{D} \\ \text{D} \end{array} \end{cases} .$$

Lemma 6.11. *Let G be an embedded graph with embedded medial graph G_m . Then if $A \subseteq E(G)$, we have*

$$Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0})^{\delta(A)}, c) = \left(\prod_{e \in A} b_e \right) Z(G; 1, \mathbf{b}_A, c, 1),$$

where $\mathbf{b}_A := \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}$.

Proof. We have

$$Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0})^{\delta(A)}, c) = \sum_{s \in \mathcal{Z}(G_m)} \omega_Z^{\delta(A)}(s) c^{c(s)} = \sum_{s \in \mathcal{Z}(G_m)} \left(\prod_{\substack{v_e \in Wh(s) \\ e \notin A}} b_e \right) \left(\prod_{\substack{v_e \notin Wh(s) \\ e \in A}} b_e \right) c^{c(s)},$$

where the sum is over all graph states s with no crossing states, and where $Wh(s)$ is the set vertices with white split states in the graph state s .

We can define a bijection between the set of embedded spanning subgraphs of G and the set of graph states of G_m by associating a spanning subgraph H_s of G by setting $e \in H_s$ if and only if the vertex $v_e \in Wh(s)$. It is then clear that for every graph state, $c(s) = f(H_s)$. By using this bijection, we can write the above state sum as

$$\left(\prod_{e \in A} b_e \right) \sum_{B \subseteq E(G)} \left(\prod_{\substack{e \in B \\ e \notin A}} b_e \right) \left(\prod_{\substack{e \in B \\ e \in A}} 1/b_e \right) c^{f(H)} = \left(\prod_{e \in A} b_e \right) Z(G; 1, \mathbf{b}_A, c, 1),$$

where $\mathbf{b}_A := \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}$ as required. \square

Our partial duality relation for $Z(G)$ given below is an extension of the duality relation in [18] and [29].

Theorem 6.12. *Let G be an embedded graph with $A \subseteq E(G)$. Then*

$$Z(G; 1, \mathbf{b}, c, 1) = \left(\prod_{e \in A} b_e \right) Z(G^{\delta(A)}; 1, \mathbf{b}_A, c, 1),$$

where $\mathbf{b} = \{b_e \mid e \in E(G)\}$ and $\mathbf{b}_A = \{b_e \mid e \notin A\} \cup \{1/b_e \mid e \in A\}$.

Proof. We have

$$\left(\prod_{e \in A} b_e \right) Z(G^{\delta(A)}; 1, \mathbf{b}_A, c, 1) = Q(G^{\delta(A)}; (\mathbf{b}, \mathbf{1}, \mathbf{0})^{\delta(A)}, c) = Q(G; (\mathbf{b}, \mathbf{1}, \mathbf{0}), c) = Z(G; 1, \mathbf{b}, c, 1),$$

where the first equality is by Lemma 6.11, the second is by Theorem 4.2 and the third follows from Proposition 5.2. \square

Remark 6.13. The signed Bollobás-Riordan polynomial was introduced by Chmutov and Pak in [10] to extend some relations between $R(G)$ and the Jones polynomial of a virtual link. It was shown to satisfy a partial duality relation in [9] (see also [30]). Vignes-Tourneret, in [33], defined a multivariate generalization of the signed Bollobás-Riordan polynomial, $Z_s(G)$, and showed that this too satisfies a partial duality relation. By Equation 3.6, Z_s can be recovered from the topochromatic polynomial $Z(G)$ and it can then be shown that Vignes-Tourneret's partial duality relation $Z_s(G)$ (Theorem 5.1 of [33]), and hence Chmutov's duality

relation for the signed Bollobás-Riordan polynomial (Theorem 3.1 of [9]) can be recovered from Theorem 6.12 above.

6.5. Twisted duals and circuits in medial graphs. In [24], Las Vergnas provided a number of formulae for enumerating Eulerian circuits of 4-regular graphs in surfaces. However, most of the formulas were given only for the sphere, torus, or projective plane. Now, with the language and tools of Section 2, we are able to extend these results to all surfaces.

We begin with the main theorem of [24] which is a formula for the number of components in a graph state without crossings of a checkerboard coloured 4-regular graph embedded in the sphere, torus, or projective plane. We note that in the language of [24], a graph state with k -components is called an Eulerian k -partition. Also, the labelling of vertex states as black or white in [24] is the reverse from that used in this paper. We have translated the terminology of [24] to be consistent with the usage in this paper.

Theorem 6.14 (Las Vergnas [24]). *Let F be a checkerboard coloured 4-regular graph embedded in the sphere, torus, or projective plane; and let s be a graph state without crossings. Then the number of circuits of s is equal to*

$$(6.1) \quad \min\{|X| + r(F_{wh}) - 2r(F_{wh}|_X) + 1, v(F) - |X| + r(F_{bl}) - 2r(F_{bl}|_Y) + 1\},$$

where X is the set of edges of F_{wh} corresponding to vertices of F with a black split in the graph state, and where Y is the set of edges of F_{bl} corresponding to vertices of F with a white split in the graph state, when we view F as the medial graph of both F_{wh} and F_{bl} .

The strength of this formula is that it computes a topological property from readily attainable quantities.

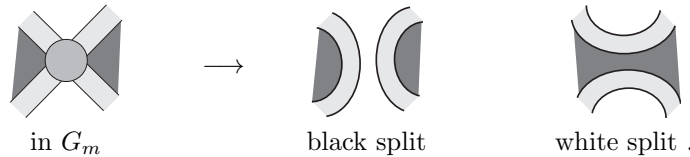
We now give a formula for the number of components of a graph state, with a much shorter proof than the original, that does hold for every surface, and use it to explain why the formula of Theorem 6.14 fails on surfaces other than the sphere, torus, or projective plane.

Proposition 6.15. *Let F be a 4-regular connected checkerboard coloured embedded graph, and let s be a graph state without crossings. Then the number of circuits in s is*

$$(6.2) \quad f(F_{bl}|_Y) = 2k(F_{bl}|_Y) - \gamma(F_{bl}|_Y) + |Y| - v(F_{bl}),$$

where Y is the set of edges of F_{bl} corresponding to vertices of F with a white split in the graph state s .

Proof. This is nearly a tautology. We see in the diagram below, an edge of F_{bl} realized as a ribbon graph together with the corresponding vertex of F , which shows that black splits essentially ‘snip through’ the corresponding edges, effectively deleting them:



Thus, the circuits of the graph state s of F_{bl} just follow the face boundaries when the edges corresponding to black splits are deleted. The number of circuits in a state with no crossings is then just $f(F_{bl}|_Y)$. The right-hand side of Equation (6.2) follows from Euler’s formula. \square

Although, since tautological, Proposition 6.15 may be less useful than Theorem 6.14, it does lead us to rewrite Theorem 6.14 in a form that reveals why it does not generalize to other surfaces.

Theorem 6.16. *Let F be a connected checkerboard coloured 4-regular graph embedded in the sphere, torus, or projective plane. Then the number of components of a graph state without crossings is equal to*

$$(6.3) \quad \min\{f(F_{bl}|_Y) + \gamma(F_{wh}|_X), f(F_{bl}|_Y) + \gamma(F_{bl}|_Y)\},$$

where X is the set of edges of either F_{bl} or F_{wh} corresponding to vertices of F with a black split in the graph state, and where Y is the set of edges of either F_{bl} or F_{wh} corresponding to vertices of F with a white split in the graph state, when we view F as the medial graph of both F_{wh} and F_{bl} .

Proof. Euler's formula states that $v(G) - e(G) + f(G) = 2k(G) - \gamma(G)$. With this,

$$\begin{aligned} |X| + r(F_{wh}) - 2r(F_{wh}|_X) + 1 &= |X| + v(F_{wh}) - 2(v(F_{wh}) - k(F_{wh}|_X)) \\ &= f(F_{wh}|_X) + \gamma(F_{wh}|_X) = f(F_{bl}|_Y) + \gamma(F_{wh}|_X), \end{aligned}$$

where the last equality follows by noting that since X and Y are complementary sets in dual graphs, $f(F_{bl}|_Y) = f(F_{wh}|_X)$. A similar calculation shows that $v(F) - |X| + r(F_{bl}) - 2r(F_{bl}|_Y) + 1 = f(F_{bl}|_Y) + \gamma(F_{bl}|_Y)$, and the result then follows by Theorem 6.14. \square

In the proof of Corollary 6.17 we can now see the importance of low genus in Theorem 6.14.

Corollary 6.17. *If F is a connected checkerboard coloured 4-regular graph embedded in the sphere, torus, or projective plane, then*

$$\min\{f(F_{bl}|_Y) + \gamma(F_{wh}|_X), f(F_{bl}|_Y) + \gamma(F_{bl}|_Y)\} = f(F_{bl}|_Y).$$

Proof. For the plane, torus, or projective plane, we note that $\gamma(F_{wh}|_Y)$ and $\gamma(F_{bl}|_X)$ are in $\{0, 1, 2\}$. For the plane, both are 0, so the result follows immediately. On the torus and the projective plane, since $(F_{wh}|_X)$ and $(F_{bl}|_Y)$ are edge disjoint (if we identify the edges of F_{wh} and F_{bl}), both cannot contain fundamental cycles. Thus, one or the other of $\gamma(F_{wh}|_X)$ and $\gamma(F_{bl}|_Y)$ must be 0, from which the result follows. This does not hold on surfaces of higher genus. \square

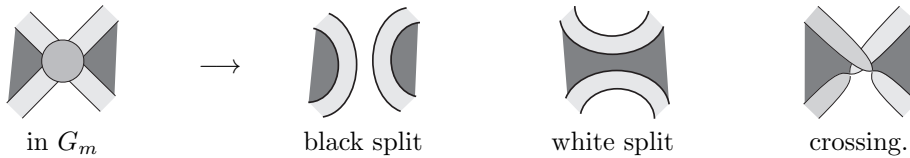
Twisted duality however, allows us to extend the enumeration formula in Proposition 6.15 to all graph states, not just those without crossings.

Proposition 6.18. *Let F be a connected checkerboard coloured 4-regular embedded graph. Then the number of circuits in any graph state is*

$$f((F_{bl})^{\tau(Z)}|_{Y \cup Z}),$$

where Y is the set of edges of F_{bl} corresponding to vertices of F with a white split in the graph state, and Z is the set corresponding to crossings.

The proof, based on the diagram below, is nearly a tautology and therefore omitted.



We provide a further illustration of how the language of embedded graphs can be used to extend and provide new perspectives on results about circuits in medial graphs by considering the following theorem of Las Vergnas. Las Vergnas provided the following application of Theorem 6.14 which relates Eulerian circuits and spanning trees.

Theorem 6.19 (Las Vergnas [24]). *Let F be a checkerboard coloured 4-regular graph embedded in the sphere, torus or projective plane. Let s be a graph state without crossings of F , let X be the set of edges of F_{wh} corresponding to vertices of F with a black split in the graph state s , and Y be the set of edges of F_{bl} corresponding to vertices of F with a white split in the graph state s . Then s defines an Euler circuit of F if and only if $F_{wh}|_X$ is a spanning tree of F_{wh} , or $F_{bl}|_Y$ is a spanning tree of F_{bl} .*

By translating this result into the language of ribbon graphs we are able to extend Theorem 6.19 to all embedded graphs. If we let G denote the whiteface graph F_{wh} , then an Eulerian circuit without crossings in F corresponds to a *pseudo-tree* of G , which is a ribbon subgraph of G that has exactly one face. In addition, $F_{wh}|_X$ corresponds to a ribbon subgraph $G - A$ of G , and $F_{bl}|_Y$ corresponds to a ribbon subgraph $G^*|_A$ of G^* , where we identify the edges of G and G^* . Thus, Theorem 6.19 is equivalent to the statement that if G is a ribbon graph homeomorphic to a punctured sphere, torus or projective plane, then $G - A$ is a pseudo-tree if and only if $G - A$ or $G^*|_A$ is a spanning tree of G . It is clear that this statement, and hence Las Vergnas' Theorem 6.19, is completed by Theorem 6.20 below.

Theorem 6.20. *Let G be a ribbon graph and $A \subseteq E(G)$. Then $G - A$ is a pseudo-tree if and only if $G^* - A^c$ is a pseudo-tree. Moreover, if $G - A$ is a pseudo-tree, then*

$$\gamma(G - A) + \gamma(G^*|_A) = \gamma(G).$$

Proof. Since the ribbon subgraphs $G - A$ and $G^*|_A$ of G have the same boundary components, $G - A$ has exactly one face if and only if $G^*|_A$ has exactly one face. This proves the first part of the theorem.

For the second statement, suppose that $G - A$ is a pseudo-tree. It then follows that G , $G - A$ and $G^*|_A$ are all connected. By Euler's formula we then have

$$\begin{aligned} \gamma(G - A) + \gamma(G^*|_A) &= e(G - A) - v(G - A) - f(G - A) + 2k(G - A) \\ &\quad + e(G^*|_A) - v(G^*|_A) - f(G^*|_A) + 2k(G^*|_A) \\ &= e(G) - v(G) - f(G) + 2 = e(G) - v(G) - f(G) + 2k(G) \end{aligned}$$

where the second equality follows since $e(G) = e(G - A) + e(G^*|_A)$, $v(G - A) = v(G)$, $v(G^*|_A) = v(G^*) = f(G)$, and $f(G - A) = f(G^*|_A) = k(G - A) = k(G^*|_A) = k(G) = 1$, as $G - A$ is a pseudo-tree. \square

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