

# A LOOP THEOREM/DEHN'S LEMMA FOR (SOME) ORBIFOLDS.

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ABSTRACT. The equivariant loop theorem implies the existence of a loop theorem/Dehn's lemma for three-orbifolds which are good (covered by a three-manifold). In this note we prove a loop theorem/Dehn's lemma for any locally orientable three-dimensional orbifold (good or bad) whose singular set is labeled with powers of 2. The proof is modeled on the standard tower construction.

We prove a version of the loop theorem for a certain class of 3-orbifolds, namely those which are locally orientable and whose singular set is labeled with powers of 2. The theorem follows easily for good orbifolds from the equivariant loop theorem [3, 1]. The novelty of the theorem below, therefore, is that it covers an infinite class of bad three-orbifolds. Our proof is modeled on the standard proof one sees in most graduate topology textbooks, which differs little from its original incarnation in the work of Papakyriakopoulos [4, 5, 6]. In particular, we have explicitly modeled our proof on that of [2]. We also prove, as an immediate result, that in a covering of an orbifold to which the loop theorem applies, orbifold incompressible 2-suborbifolds lift to orbifold incompressible 2-suborbifolds.

We begin with some basic notions and technical results, all of which are straight-forward generalizations of concepts common in 3-manifold theory. We denote by  $|Q|$  the underlying space of an orbifold  $Q$ , which will be a manifold when  $Q$  is locally orientable. Also when  $Q$  is locally orientable, the orbifold boundary  $\partial Q$  of  $Q$  coincides with the boundary of the underlying manifold, and we use the same notation for both. The singular set of  $Q$  is denoted  $\Sigma_Q$ , or just  $\Sigma$ , depending on the context. We let  $D(n)$  denote the quotient of the disc  $D^2 = \{z \in \mathbb{C} \mid |z| \leq 1\}$  by the rotation  $z \mapsto z^n$ .

**Definitions.** In a simplicial 3-orbifold  $Q$ , we define the *orbifold regular neighborhood* of a subcomplex  $A$  to be a subcomplex  $N(A)$  of  $Q$  that strong deformation retracts onto  $(N(A) \cap \Sigma) \cup A$ . As usual, the simplices intersecting  $A$  in the second barycentric subdivision provide the prototype for an orbifold regular neighborhood of  $A$ .

An *orbifold embedding* of an orbifold  $X$  into  $Q$  is a label-preserving map  $f: (X, \Sigma_X) \rightarrow (Q, \Sigma_Q)$  so that the associated map  $|X| \rightarrow |Q|$  is an embedding transverse to  $\Sigma_Q$ . An orbifold embedding is *proper* if  $f^{-1}(\partial Q) = \partial X$ . A two-sided 2-suborbifold  $F$  in a locally orientable 3-orbifold  $Q$  is *orbifold compressible* if there is an orbifold embedded disc  $(D(n), \partial D(n)) \rightarrow (Q, F)$

with  $[\partial D(n)]^n$  non-trivial in  $\pi_1^{\text{orb}}(F)$ . Otherwise,  $F$  is *orbifold incompressible*. Such a disc is an *orbifold compressing disc*.

In the case  $F \neq \mathbb{RP}^2$  and  $Q$  is good,  $[\partial D(n)]^n$  non-trivial in  $\pi_1^{\text{orb}}(F)$  is equivalent to saying that  $\partial D(n)$  does not bound an orbifold disc (with singular point labeled  $n$ ) in  $F$ .

A loop  $\gamma: S^1 \rightarrow Q \setminus \Sigma$  is *orbifold null-homotopic* if it lifts to the universal orbifold cover  $\tilde{Q}$ . A map  $f: D^2 \rightarrow Q$  is a *wound disc* if  $f(D^2) \cap \Sigma$  is a finite set of points  $x_1, \dots, x_k$  in the interiors of edges of  $\Sigma$  labeled  $n_1, \dots, n_k$ , respectively, at which  $f$  is transverse to  $\Sigma$  and such that, for each  $i$ ,  $f^{-1}(x_i)$  is a single point  $p_i$  contained in an open neighborhood  $U_i \subset D^2$  on which  $f$  acts as the map  $z \mapsto z^{n_i}$ . In other words, for each  $x_i \in f(D^2) \cap \Sigma$ , there exist homeomorphisms  $(U_i, p_i) \cong (D^2, 0)$  and  $(f(U_i), x_i) \cong (D^2, 0)$  such that the following diagram commutes:

$$\begin{array}{ccc} D^2 & \xrightarrow{z \mapsto z^n} & D^2 \\ \cong \downarrow & & \downarrow \cong \\ U_i & \xrightarrow{f|_{U_i}} & f(U_i) \end{array}$$

An *orbifold null-homotopy* of a loop  $\gamma$  is a wound disc with  $f|_{\partial D^2} = \gamma$ .

**Proposition.** *A loop  $\gamma$  is orbifold null-homotopic in  $Q$  if and only if there is an orbifold null-homotopy of  $\gamma$  in  $Q$ .*

*Proof.* Suppose  $\gamma$  is orbifold null-homotopic in  $Q$  and let  $\tilde{Q}$  be the universal orbifold cover of  $Q$ . If we can find a wound disc  $f: D^2 \rightarrow \tilde{Q}$ , then we can homotope  $f$  so that each winding point of  $f(D^2)$  has a neighborhood equivariant under the local group actions corresponding to the covering  $p: \tilde{Q} \rightarrow Q$ . It follows that the property of being wound is preserved under composition with orbifold covering projections. It therefore suffices to find a wound disc in  $\tilde{Q}$  whose boundary projects to  $\gamma$ .

If  $\tilde{Q}$  has non-empty singular set  $\Sigma_{\tilde{Q}}$ , then the fundamental group of the complement  $\tilde{Q} \setminus \Sigma_{\tilde{Q}}$  is normally generated by paths  $\alpha_i = \delta_i^{-1} \lambda_i \delta_i$ , where  $\lambda_i$  is a small loop running once around a singular arc labeled  $n_i$ , and  $\delta_i$  is a path running from the basepoint to  $\lambda_i$ . Adding in the relations  $\alpha_i^{n_i} = 1$  makes  $\tilde{\gamma}$  trivial. It follows that the class represented by  $\tilde{\gamma}$  is contained in the normal subgroup of  $\pi_1(\tilde{Q} \setminus \Sigma_{\tilde{Q}})$  generated by the terms  $\alpha_i^{n_i}$ , so we may represent  $[\tilde{\gamma}]$  as a product of the form  $w_0 A_{i_1} w_1 A_{i_2} \cdots A_{i_t} w_t$ , where  $A_i = \delta_i^{-1} \lambda_i^{n_i} \delta_i$  and  $w_0 w_1 \cdots w_t$  is trivial in  $\pi_1(\tilde{Q} \setminus \Sigma_{\tilde{Q}})$ .

Note that a map of a circle onto a path representing  $A_i$  extends over a disc, by first ‘‘collapsing’’ the parts of the circle mapping to  $\delta_i^{\pm 1}$ , and then extending over the remaining disc by the map  $z \mapsto z^{n_i}$ . Doing this for each  $A_i$  leaves a loop representing  $w_0 w_1 \cdots w_t$  and thus bounding a disc in  $\tilde{Q} \setminus \Sigma_{\tilde{Q}}$ . There is therefore a wound disc in  $\tilde{Q}$  with boundary  $\tilde{\gamma}$ .

Conversely, suppose that we have a wound disc  $f: D^2 \rightarrow Q$ . If  $\sigma_i$  is a loop about  $f^{-1}(x_i)$ , then  $f(\sigma)$  is a loop about  $x_i$  traversed  $n_i$  times. Such loops lift to  $\tilde{Q}$ . Since these  $\sigma_i$  generate  $\pi_1(D^2 - f^{-1}(\Sigma))$ , it follows that  $f|_{\partial D^2}$  is null-homotopic in  $Q$ .  $\square$

**Homotopy Lifting Lemma.** *Given an orbifold null-homotopy  $f: D^2 \rightarrow Q$  of a loop  $\gamma$ , an orbifold covering  $p: \hat{Q} \rightarrow Q$ , and a lift  $\hat{\gamma} \subset \hat{Q}$  of  $\gamma$ , there is a map  $\hat{f}: D^2 \rightarrow \hat{Q}$  such that  $p \circ \hat{f} = f$  and  $\hat{f}(\partial D^2) = \hat{\gamma}$ .*

*Proof.* Let  $f$ ,  $\gamma$ ,  $p$ , and  $\hat{\gamma}$  be as in the statement, and let  $x_1, \dots, x_k$  and  $n_1, \dots, n_k$  be as in the proposition above. That  $f$  lifts to a map  $\hat{f}: D^2 - \{x_1, \dots, x_k\} \rightarrow \hat{Q} - \hat{\Sigma}$  follows as in the last paragraph of the proof above.

Now consider a point  $f(x_i) \in \Sigma$ , and let  $d = n_i/m$ . We need only extend the lift  $\hat{f}$  continuously to  $x_i$ . To this end, let  $B^3 = \{(z, x) \in \mathbb{C} \times \mathbb{R} \mid |z|^2 + x^2 < 1\}$ . Then on a neighborhood  $U_i$  of  $x_i$  we have the following:

$$\begin{array}{ccccccc} D^2 & \xrightarrow{z \mapsto z^{n_i}} & D^2 & \xrightarrow{i} & B^3 & \xleftarrow{(z^d, x) \leftarrow (z, x)} & B^3 \\ h_1 \downarrow & & \cong \downarrow & & \downarrow \cong & & \downarrow h_2 \\ U_i & \xrightarrow{f|_{U_i}} & f(U_i) & \xrightarrow{i} & V_i & \xleftarrow{p} & \hat{V}_i \end{array}$$

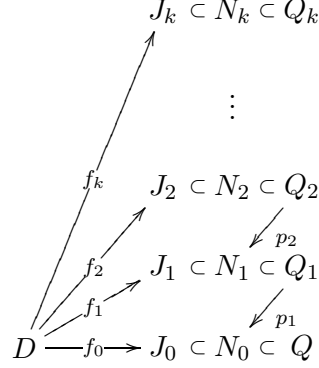
where all of the vertical maps are homeomorphisms, and  $V_i$  is a neighborhood of  $f(x_i)$  in  $Q$ . Note that the map  $g(z) = (z^{n_i/d}, 0)$  along the top row is well-defined, since  $d|n_i$ . We may thus define  $\hat{f}: U_i \rightarrow \hat{V}_i$  to be  $\hat{f} = h_2 \circ g \circ h_1^{-1}$ . Moreover, this lift agrees (up to isotopy) with the previously defined  $\hat{f}$ , which is unique (up to isotopy). We may therefore extend  $\hat{f}$  over all of  $D^2$ .  $\square$

**Theorem.** *Let  $Q$  be a locally orientable 3-orbifold with all singular arcs labeled with powers of 2, and suppose  $K = \text{Ker}\{i_*: \pi_1^{\text{orb}}(\partial Q) \rightarrow \pi_1^{\text{orb}}(Q)\}$  is non-trivial. Let  $N \subset K$  be a proper normal subgroup of  $\pi_1^{\text{orb}}(\partial Q)$ . Then there is a properly embedded orbifold disc  $D^2(n)$  in  $Q$  with the property that  $[\partial D^2(n)]^n \notin N$ .*

*Proof.* Since  $K$  is non-trivial, there is a null-homotopy  $f: (D^2, \partial D^2) \rightarrow (Q, \partial Q)$  with  $[f(\partial D^2)] \neq 1 \in \pi_1^{\text{orb}}(\partial Q)$ . We may triangulate  $Q$  and  $D^2$  so that the singular set  $\Sigma$  of  $Q$  is a subcomplex and  $f$  is homotopic to a simplicial map  $f_0$ .

Let  $N_0$  be an orbifold regular neighborhood of the image  $J_0$  of  $f_0$ . Suppose  $N_0$  has a (connected) two-fold orbifold cover  $p_1: Q_1 \rightarrow N_0$ . Note that  $p_1^{-1}(J_0)$  is connected, since  $N_0$  retracts onto  $J_0$ , and we can lift this retraction to one of  $Q_1$  onto  $p_1^{-1}(J_0)$ .

By the homotopy lifting lemma,  $f_0$  lifts to a map  $f_1: D^2 \rightarrow Q_1$  whose image  $J_1$  has orbifold regular neighborhood  $N_1$ . Continue, as long as possible, to take orbifold regular neighborhoods and orbifold double covers. We will show that this process terminates, so that for some  $k$ ,  $N_k$  has no orbifold double cover.



To this end, consider a typical orbifold double cover  $p_i: Q_i \rightarrow N_{i-1}$ , and let  $\rho$  be the corresponding orbifold covering transformation. Then  $J_i \cap \rho(J_i) \neq \emptyset$ , since  $J_i \cup \rho(J_i) = p_i^{-1}(J_{i-1})$  is connected. (Note that we may have  $J_i = \rho(J_i)$ .) If  $\rho$  has no fixed points in  $J_i$ , then there are two simplices in the intersection with disjoint interiors that are interchanged by  $\rho$ . In passing to  $J_{i-1}$ , these simplices are identified. Hence,  $J_i$  has strictly more simplices than  $J_{i-1}$ .

On the other hand, suppose  $\rho$  has a fixed point  $x \in J_i$ . Then in a 3-ball neighborhood of  $x$ ,  $\rho$  acts as rotation by  $\pi$  about an axis passing through  $x$  and transverse to both  $J_i$  and  $\rho(J_i)$ . It follows that  $J_i \cap \rho(J_i)$  contains (at least) two edges sharing the vertex  $x$ . These are interchanged by  $\rho$ , from which we deduce, as before, that  $J_i$  has strictly more simplices than  $J_{i-1}$ . The claim follows from the fact that the number of simplices in any  $J_i$  is bounded above by the number of simplices in the triangulation on  $D^2$ .

We have now an orbifold  $N_k$  that is a regular neighborhood of a mapped-in disc and that has no two-fold orbifold covers. Note that this implies that the underlying manifold has no two-fold covers, so  $\text{Hom}(H_1|N_k|, \mathbb{Z}_2) = \{0\}$ . Using Poincaré-Lefschetz duality, universal coefficients, and dual vector spaces, we deduce that in the following piece of a long exact sequence:

$$H_2(|N_k|, |\partial N_k|; \mathbb{Z}_2) \rightarrow H_1(|\partial N_k|; \mathbb{Z}_2) \rightarrow H_1(|N_k|; \mathbb{Z}_2)$$

the outermost terms are both zero. Thus  $H_1(|\partial N_k|; \mathbb{Z}_2) = \{0\}$ , which implies that  $N_k$  has boundary components that are topological spheres.

Let  $A$  be the component of  $\partial N_k$  on which  $f_k(\partial D)$  lies, and let  $A' = (p_1 \circ \cdots \circ p_k)^{-1}(\partial Q) \subset A$ . Note that  $A'$  contains no singular points, since the projection of  $A'$  into  $\partial Q$  is an orbifold regular neighborhood of a closed loop in  $\partial Q$  avoiding singular points. Thus  $A'$  is a non-singular planar surface with (orbifold) fundamental group generated by simple closed curves. Now  $f_k(\partial D)$  is a word in these generators with the property that its projection into  $\pi_1^{\text{orb}}(\partial Q)$  is not contained in  $N$ . It follows that some generator also has this property.

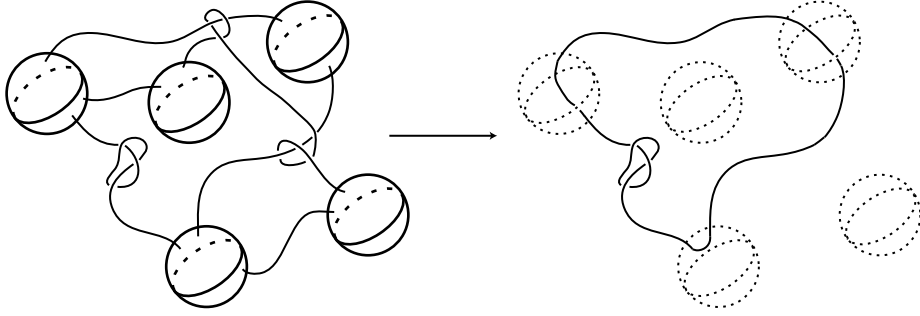


FIGURE 1. Obtaining a single loop labeled 2

Unlike the manifold case, it is not immediately clear that such a curve bounds an embedded orbifold disc in  $N_k$  (there may be more than one singular arc passing through each of the two disc components of  $A$  bounded by the curve). It is not even obvious that we may choose this simple closed curve to represent a trivial element in  $\pi_1^{\text{orb}}(N_k)$ , let alone to have its projection contained in  $K$ . In order to get such an embedded orbifold disc, we use the hypothesis on the labels of the singular set of  $Q$ .

Let  $\Gamma$  be a graph containing one vertex for each boundary component of  $N_k$ , and with the property that two vertices are connected by one edge for each singular arc in  $N_k$  joining the two corresponding boundary components. Suppose (for contradiction) this graph contains a circuit, and choose one with a minimal number of vertices. Then we construct an orbifold  $\mathcal{O}$  from  $N_k$  by changing edge labels and capping boundary components as follows: Label with a 2 every singular arc in  $N_k$  corresponding to an edge in the circuit, make all other arcs non-singular, and cap off all boundary components with balls. In the balls attached to those boundary components corresponding to vertices in the circuit, insert a single arc labeled 2 joining the two singular arcs intersecting the boundary of the ball (there are only two by the minimality assumption above). Then  $\mathcal{O}$  is a closed orbifold containing a single (possibly knotted) loop  $\ell$  labeled 2 (see Figure 1).

Note that  $|\mathcal{O}|$  is obtained from  $|N_k|$  by capping off sphere boundary components with balls. This has no effect on first homology, so it follows that  $H_1(|\mathcal{O}|; \mathbb{Z}_2) = H_1(|N_k|; \mathbb{Z}_2) = 0$ . In particular,  $\ell$  is null-homologous (with  $\mathbb{Z}_2$  coefficients) so there is a well-defined map  $h: \pi_1(|\mathcal{O}| - \ell) \rightarrow \mathbb{Z}_2$  where  $h([\gamma])$  is the mod-2 linking number of  $\gamma$  with  $\ell$ . This implies that  $|\mathcal{O}| - \ell$  has a two-fold cover that unwraps around  $\ell$ . Since  $\ell$  is labeled 2, this cover induces a two-fold orbifold cover  $\tilde{\mathcal{O}}$  of  $\mathcal{O}$ .

Now since the balls glued onto the boundary components of  $N_k$  lift to balls in the cover, we may remove them from  $\mathcal{O}$  while preserving the orbifold cover on the remaining orbifold. Once this is done, we observe that the orbifold covering projection is locally a homeomorphism away from the lift of  $\ell$ . We

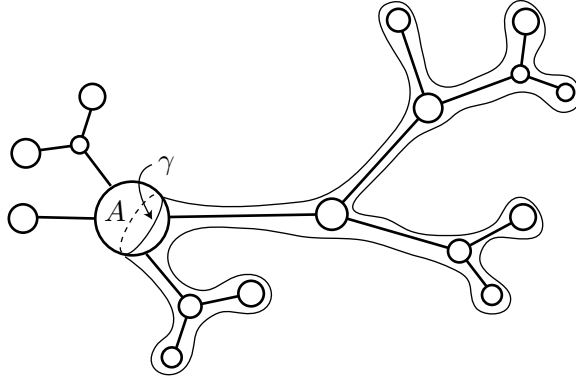


FIGURE 2. An embedded disc with boundary  $\gamma$

may therefore restore the labels on the singular set not in  $\ell$ , placing the same labels on the lifts, and still have an orbifold cover. Finally, since each component of  $\Sigma$  giving rise to a part of  $\ell$  has label divisible by two, we may restore the labels on these components, giving the lifted components half the label of their projections. We thus obtain a two-fold orbifold cover of  $N_k$ , contrary to assumption, and deduce that the singular set of  $N_k$  contains no circuits.

It follows from this that any simple closed curve  $\gamma$  in  $A$  bounds an embedded non-singular disc in  $N_k$  (see Figure 2).

We now show that we may obtain orbifold compressing discs in each  $N_i$  by pushing this disc down the tower. Assume we have a properly embedded orbifold compressing disc  $S$  in  $N_i$ . Suppose also that  $S$  is transverse to its covering translate  $\rho S$ . Let  $T = p_i(S)$ . Since the restriction of  $p_i$  to the complement of  $\text{Fix}(\rho)$  is an ordinary two-fold cover, we may assume that  $T$  has, at worst, double arcs, coming from intersections of  $S$  with  $\rho S$ . The simple closed curves and properly embedded arcs in  $S \cap \rho S$  may contain isolated points of intersection with  $\text{Fix}(\rho)$ , subject to the following restrictions:

- (1) any point in  $\text{Fix}(\rho)$  is in the interior of exactly one curve in  $S \cap \rho S$  (since  $S$  and  $\rho S$  are transverse),
- (2) there is at most one fixed point along any properly embedded arc in  $S \cap \rho S$  (since any homeomorphism of an interval to itself with two or more fixed points is orientation preserving, while  $\rho$  is locally rotation by  $\pi$  around its fixed points), and
- (3) any simple closed curve in  $S \cap \rho S$  contains zero or two fixed points (similar to (2)).

We will first attempt to make  $S$  and  $\rho S$  disjoint from  $\text{Fix}(\rho)$ . In the one case that this is impossible, we may at least make  $S$  intersect  $\text{Fix}(\rho)$  only in its singular point, so that in either case  $T$  has at most one singular point.

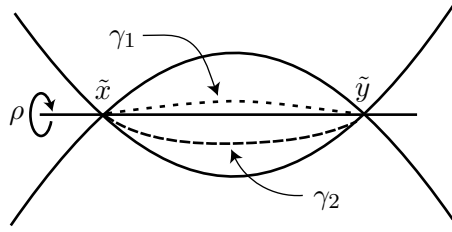


FIGURE 3. A simple closed curve intersecting  $\text{Fix}(\rho)$  twice.

We will then adjust  $T$  directly, in order to produce the desired embedded orbifold compressing disc.

Intersections of  $S$  and  $\rho S$  with  $\text{Fix}(\rho)$  come in two types: simple closed curves in  $S \cap \rho S$  intersecting  $\text{Fix}(\rho)$  in two points and properly embedded arcs in  $S \cap \rho S$  intersecting  $\text{Fix}(\rho)$  in one point. We begin by eliminating the former. Choose a double arc  $\gamma$  in  $T$  with two singular points  $x$  and  $y$  in its closure. Let  $\tilde{x}$  and  $\tilde{y}$  be the lifts of these points in  $N_i$ , and let  $\gamma_1$  and  $\gamma_2$  be the two lifts of  $\gamma$  joining  $\tilde{x}$  and  $\tilde{y}$ . Note that  $\gamma_1 \cup \gamma_2$  bounds a disc  $D_S$  in  $S$  and a disc  $D_{\rho S}$  in  $\rho S$ . Assume that  $D_S$  is innermost in  $S$ , and note that this implies that  $D_{\rho S}$  is innermost in  $\rho S$ .

Now if we remove  $D_S$  from  $S$  and replace it with  $D_{\rho S}$ , and vice versa, and then equivariantly push off  $\text{Fix}(\rho)$ , we get two new discs still satisfying all of the necessary hypotheses but with fewer intersections with  $\text{Fix}(\rho)$ . Continuing this process, we may remove all remaining circles of intersection which pass through  $\text{Fix}(\rho)$ . For simplicity, we will refer to the resulting discs as  $S$  and  $\rho S$ , and will continue to do so as long as it still makes sense (i.e., as long as our operations are performed equivariantly on  $S$  and  $\rho S$  — we will later need to alter  $T$  directly in ways that cannot be described via alterations in the covering space).

To finish making  $S$  and  $\rho S$  disjoint from  $\text{Fix}(\rho)$ , we need just remove properly embedded arcs of intersection which pass through  $\text{Fix}(\rho)$ . Such an arc splits the boundary of  $S$  into two pieces, which we label  $a$  and  $b$ , coherently oriented. We will adjust  $S$  and  $\rho S$  in a neighborhood of the arc in such a way that each new disc will consist of one half of  $S$  and one half of  $\rho S$ . Note that there are two ways of doing this (see Figure 4), both of which alter the boundary, with the new boundary equal either to  $a^{-1}b$  or to  $a^2$ . We will show that one of the two resulting pairs of discs still has (the appropriate power of) its boundary mapping into  $K \setminus N$ .

Suppose that  $S$  has singular point labeled  $n$  disjoint from  $\text{Fix}(\rho)$ , and assume it lies on the  $b$  side of the arc  $S \cap \rho S$ . Then after cutting and pasting as described above,  $S$  either has boundary  $a^{-1}b$  and singular point labeled  $n$  or has boundary  $a^2$  and no singular point. Thus we need either  $(a^{-1}b)^n$

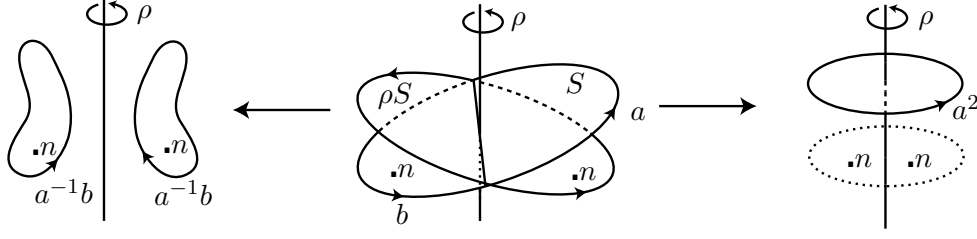


FIGURE 4. An embedded arc of  $S \cap \rho S$  intersecting  $\text{Fix}(\rho)$ .

or  $a^2$  to project into  $K \setminus N$ . But note that

$$(ab)^n = [(ab)a^2]^n (a^{-1}b)^n [(ab)^{-1}]^n,$$

so if both  $(a^{-1}b)^n$  and  $a^2$  map into the normal subgroup  $N$ , then so does  $(ab)^n$ , contrary to assumption. It follows that we may remove this component of  $S \cap \rho S$  by one of the two alterations discussed.

In case there is no singular point on  $S$  away from  $\text{Fix}(\rho)$ , we have, after cutting and pasting, that  $S$  either has boundary  $a^{-1}b$  and no singular point or has boundary  $a^2$  and singular point labeled  $n$ . Thus we need either  $a^{-1}b$  or  $a^{2n}$  to project into  $K \setminus N$ . But note that

$$(ab)^n = (ab)^n a^{2n} [a^{-1}b(ab)^{-1}]^n,$$

so that, as above, one of the two alterations produces a disc with boundary projecting into  $K \setminus N$ .

We may now assume that double curves in  $T$  consist of simple closed curves and properly embedded arcs, all of which are disjoint from the singular set. Hereafter, we will need to alter  $T$  directly, rather than indirectly via  $S$ .

To begin the process of removing the double curves of  $T$ , we first remove the simple closed curves. We do this exactly as in the manifold case, which we now sketch. For more details, see [2]. Let  $C$  be a double curve in  $N_{i-1}$ , and let  $C_1$  and  $C_2$  be the two components of  $p_i^{-1}(C)$ . A neighborhood of  $C$  in  $T$  is an  $I$ -bundle over an  $X$  shape.

Assume first the bundle is trivial. Then if  $C_1$  and  $C_2$  cobound an annular region in  $S$ , we remove the image of this region from  $T$ , smoothing along the corner, thereby removing the double curve  $C$ . Otherwise,  $C_1$  and  $C_2$  bound distinct transverse discs, and we may exchange the images of these discs.

Now suppose the bundle is non-trivial. It must then be  $X \times I$  with ends identified by a vertical (say) reflection. This is an immersed annulus, which we replace by an embedded annulus with the same boundary by splitting the top of the  $X$  from the bottom. In this manner, we may remove all remaining double closed curves.

Note that we may, in this process, remove the singular point from  $T$ . In this case, we no longer need for  $(\partial T)^n$  to project into  $K \setminus N$ , but rather

for  $\partial T$  to do so. But since the new  $T$  has no singular point, it is clear that  $\partial T$  maps into  $K$ , and the fact that some power of  $\partial T$  projects into  $K \setminus N$  implies that  $\partial T$  does also.

We now assume that the double arcs of  $T$  are all properly embedded arcs, disjoint from the image of  $\text{Fix}(\rho)$ . We will think of  $T$  as a disc with identifications. There are two ways in which two arcs in  $T$  can be identified. We will call these type (i) and type (ii) identifications, as shown in Figure 5. We require different treatments for the different ways in which a singular point may be situated on  $T$  vis-à-vis the identified arcs. The situation in which there is a singular point between the two arcs is similar to that in which there is no singular point at all, in which case we say  $T$  is of type 1. Otherwise there is a singular point elsewhere on  $S$  labeled  $n > 1$ , and we say  $T$  is of type 2. To simplify the discussion, we will refer to the case that

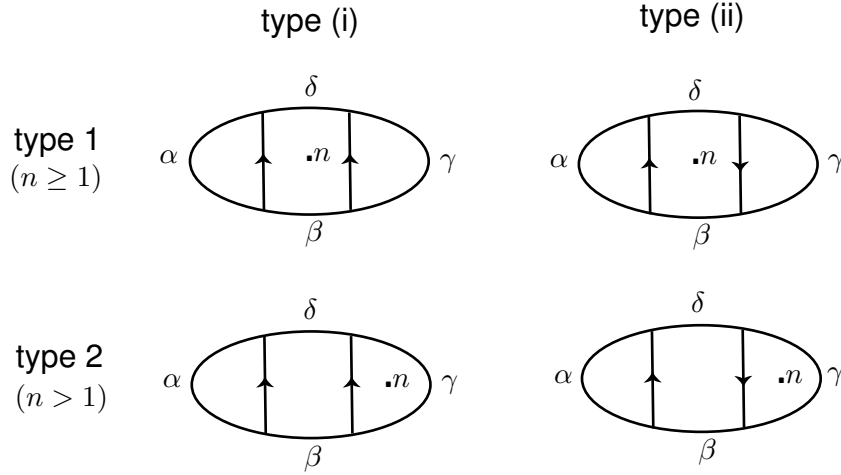


FIGURE 5. Four cases of identified non-singular arcs in  $T$ .

$T$  is type 1 and the identification is of type (i) as case 1(i), etc.

In all cases, we will consider the boundary of  $T$  to consist of four coherently oriented arcs  $\alpha, \beta, \gamma, \delta$ , as in figure 5. Also, we will in each case alter  $T$  in one of two ways to remove a double arc, both of change the boundary. As before, it will be the case that one of the two new boundaries (or, rather, the appropriate power of such a boundary) continues to project into  $K \setminus N$ .

As cases 1(i) and 1(ii) include the cases that  $T$  is non-singular, the procedure involved is identical to that used in the loop theorem for manifolds, with some minor additional verification involved to take care of the possible singular point on  $T$ . In cases 1(i) and 1(ii), we cut and paste as in Figure 6. In case 1(i), we get a disc with boundary either  $\alpha\gamma$  or  $\alpha\beta^{-1}\gamma\delta^{-1}$ . In case 1(ii), we get a disc with boundary either  $\alpha\gamma^{-1}$  or  $\alpha\delta\gamma\beta$ . We then observe

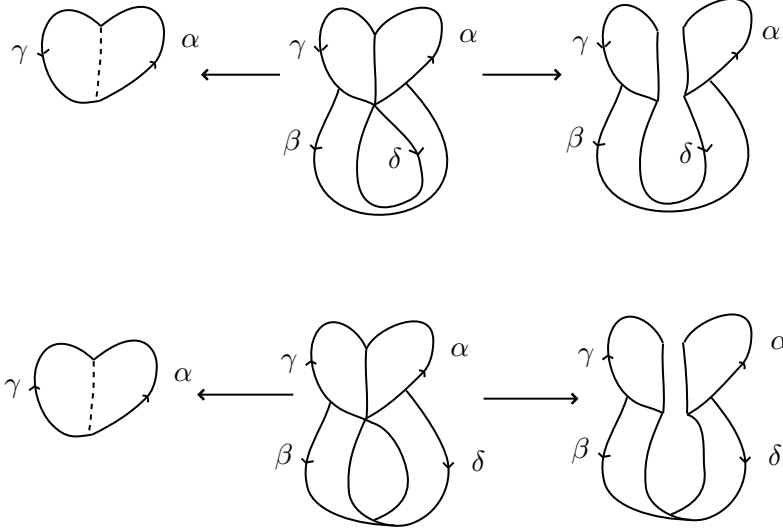


FIGURE 6. Case 1: When there is a singular point between the identified arcs, or none at all.

that both

$$\delta^{-1}[(\alpha\beta\alpha^{-1})^{-1}(\alpha\gamma)\delta^{-1}]^n(\alpha\beta^{-1}\gamma\delta^{-1})^{-n}[\delta(\alpha\beta\alpha^{-1})(\alpha\gamma)]^n\delta$$

and

$$(\gamma\delta)^{-1}[(\alpha\beta)^{-1}(\alpha\gamma^{-1})(\gamma\delta)^{-1}(\alpha\gamma^{-1})^{-1}]^n(\alpha\delta\gamma\beta)^n[(\gamma\delta)(\alpha\beta)]^n(\gamma\delta)$$

are equal to  $(\alpha\beta\gamma\delta)^n$ , for  $n \geq 1$ . So in either case, there is a product of conjugates of the two possible boundaries projecting outside of the normal subgroup  $N$ . It therefore cannot be that both possible boundaries project into  $N$ . We now replace  $T$  with the altered disc having boundary projecting into  $K \setminus N$ , and note that this new disc has (at least) one less double arc than did  $T$ .

For cases 2(i) and 2(ii), we assume that the identified arc in  $T$  farther from the singular point is outermost among all identified arcs. In these cases, we consider a different option for altering  $T$ . We will assume the singular point lies on the  $\gamma$  side of the identified arcs in  $T$ . In each case, for one of the two possible new discs, we still consider that with boundary  $\alpha\gamma$  (in case 2(i)) or  $\alpha\gamma^{-1}$  (in case 2(ii)). For the other option, we consider a disc with boundary isotopic  $\alpha\beta\alpha^{-1}\delta$  (in case 2(i)) or  $\alpha\delta\alpha\beta$  (in case 2(ii)), making use of a parallel copy of the projection of the (non-singular) piece of  $T$  with  $\alpha$  in its boundary; see Figure 7.

We now note that both

$$\delta^{-1}[(\alpha\beta\gamma)^{-1}(\alpha\beta\alpha^{-1}\delta)\delta^{-1}]^n(\alpha\gamma)^n[\delta(\alpha\beta\gamma)]^n\delta$$

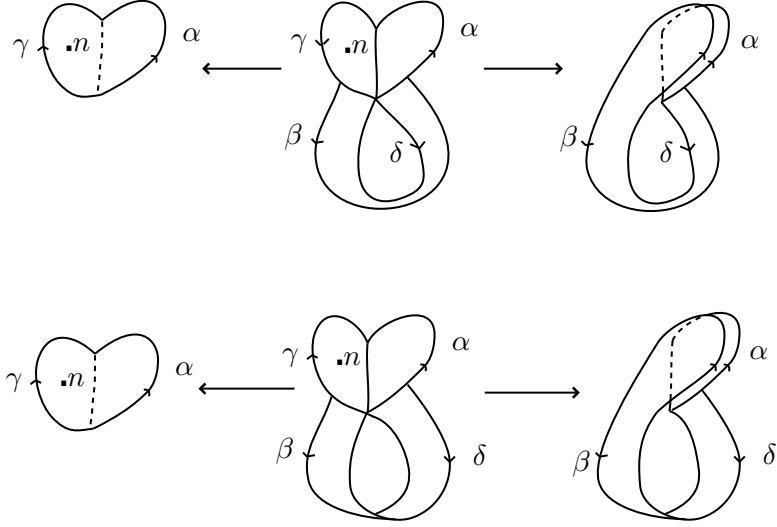


FIGURE 7. Case 2: When there is a singular point not between the identified arcs.

and

$$(\gamma\delta)^{-1}[(\gamma\delta\alpha\beta)^{-1}(\alpha\gamma^{-1})^{-1}]^n(\alpha\delta\alpha\beta)^n(\gamma\delta\alpha\beta)^n(\gamma\delta)$$

are equal to  $(\alpha\beta\gamma\delta)^n$ , which implies, as above, that in both case 2(i) and case 2(ii), there is a choice of alterations of  $T$  that results in a disc having (at least) one less double arc and whose boundary projects into  $K \setminus N$ .

Repeating this process, we remove all remaining double arcs, thereby producing an embedded orbifold disc in  $N_{i-1}$  whose boundary maps into  $K \setminus N$ , moving us one level down the tower. Continuing to the case  $i = 1$  completes the proof.  $\square$

As in the manifold case, we have the following corollary.

**Corollary.** *Let  $Q$  be a 3-orbifold for which the conclusion of the loop theorem holds (e.g.,  $Q$  is good, or has singular set labeled with powers of 2). Let  $F$  be a two-sided orbifold incompressible 2-suborbifold of  $Q$ , and let  $p: \widehat{Q} \rightarrow Q$  be a finite orbifold cover. Then  $p^{-1}(F)$  is orbifold incompressible in  $\widehat{Q}$ .*

*Proof.* Suppose  $p^{-1}(F)$  is orbifold compressible, and let  $f: D^2 \rightarrow \widehat{Q}$  be an orbifold compressing disc with  $f(\partial D^2) = \gamma \subset p^{-1}(F)$ . We may assume that all circles in  $f(D^2) \cap p^{-1}(F)$  are trivial in  $\pi_1^{\text{orb}}(p^{-1}(F))$ , by passing to innermost (in  $D^2$ ) essential (in  $\pi_1^{\text{orb}}(p^{-1}(F))$ ) circles. Then each circle in  $f(D^2) \cap p^{-1}(F)$  bounds a disc in  $p^{-1}(F)$ . Cut and paste along each of these circles (beginning with those innermost in  $D^2$ ), removing the intersections by replacing each portion of  $f(D^2)$  bounded by such a circle with a parallel copy of the disc in  $p^{-1}(F)$  the circle bounds. Since the boundary has not changed, we now have an orbifold compressing disc  $f: D^2 \rightarrow \widehat{Q}$  with  $f(D^2) \cap p^{-1}(F) =$

$f(\partial D^2)$ . Composing with  $p$ , we obtain a wound disc  $(p \circ f): D^2 \rightarrow Q$  with  $(p \circ f)(D^2) \cap F = (p \circ f)(\partial D^2)$ . Splitting  $Q$  open along  $F$ , we obtain an orbifold  $Q'$  containing a wound disc  $g: D^2 \rightarrow Q'$  with  $g(\partial D^2) \subset F$ . The loop theorem then provides an orbifold compressing disc with boundary in  $F$ , and the result follows by gluing  $Q$  back together.  $\square$

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