

Solutions to practice problems for Calculus I Test I.

1. Write the precise ϵ - δ definition of the following: $\lim_{g \rightarrow f} x(g) = a$.

$\lim_{g \rightarrow f} x(g) = a$ means that for all $\epsilon > 0$ there exists a $\delta > 0$ so that $|x(g) - a| < \epsilon$ whenever $|g - f| < \delta$.

2. Illustrate the definition of $\lim_{x \rightarrow 4} (3x - 2) = 10$ by finding a value of δ that corresponds to $\epsilon = 1/2$. Justify your choice.

We want to have $|3x - 2 - 10| < 1/2$. This is equivalent to wanting to have $3|x - 4| < 1/2$, or $|x - 4| < 1/6$, so we choose $\delta = 1/6$.

3. Use the definition of limit to prove that $\lim_{x \rightarrow 0^+} \sqrt{x} = 0$.

Fix $\epsilon > 0$ and let $\delta = \epsilon^2$ (which we found by a process similar to that above). Then $|x - 0| < \delta$ is equivalent to $|x - 0| < \epsilon^2$, which is the same as $|\sqrt{x}| < \epsilon$, or in other words $|\sqrt{x} - 0| < \epsilon$.

4. Sketch the graphs (on a single xy -plane) of the functions $y = x^{1/3}$, $y = x^{1/4}$ and $y = x^{1/5}$, indicating which curve corresponds to which function.

Both $x^{1/3}$ and $x^{1/5}$ should look like x^3 but reflected left/right and then rotated clockwise 90° , and with $x^{1/5}$ slightly skinnier than $x^{1/3}$. The graph for $x^{1/4}$ looks like the left side of a parabola rotated 90° clockwise, and it should lie between the corresponding parts of $x^{1/3}$ and $x^{1/5}$.

5. For the functions $f(x) = \frac{1}{x}$ and $g(x) = x^3 + 3x$, calculate the compositions $f \circ g$ and $g \circ f$ and find the domain of each.

$$(f \circ g)(x) = f(g(x)) = f(x^3 + 3x) = \frac{1}{x^3 + 3x}$$

The domain is all x for which the denominator is not zero; i.e., all x except $x = 0$.

$$(g \circ f)(x) = g(f(x)) = g(1/x) = \frac{1}{x^3} + \frac{3}{x}$$

As above, the domain is all x except $x = 0$.

6. Calculate the limit $\lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4}$.

$$\begin{aligned} \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} &= \lim_{x \rightarrow 4} \frac{\sqrt{x} - 2}{x - 4} \cdot \frac{\sqrt{x} + 2}{\sqrt{x} + 2} = \\ \lim_{x \rightarrow 4} \frac{x - 4}{(x - 4)(\sqrt{x} + 2)} &= \lim_{x \rightarrow 4} \frac{1}{\sqrt{x} + 2} = \frac{1}{4} \end{aligned}$$

7. Does $\lim_{x \rightarrow 0} (x \sin(1/x))$ exist? Why or why not?

Note that $-x \leq x \sin(1/x) \leq x$ for all x . Also, $\lim_{x \rightarrow 0} -x = 0 = \lim_{x \rightarrow 0} x$, so the squeeze theorem tells us that $\lim_{x \rightarrow 0} (x \sin(1/x))$ exists and equals 0.

8. State the intermediate value theorem.

Suppose $f(x)$ is a function continuous on $[a, b]$ with $f(a) < f(b)$, and choose a number N with $f(a) < N < f(b)$. Then there is some number c with $a < c < b$ so that $f(c) = N$.

9. Show that the equation $x^5 - x^2 - 4 = 0$ has at least one solution.

Let $f(x) = x^5 - x^2 - 4$, and note that $f(1) = -1 < 0$ while $f(2) = 24 > 0$. Thus by the intermediate value theorem there is some number c between 1 and 2 so that $f(c) = 0$.

10. Let $f(x)$ be the function in problem #3 on page 46 (the blue one). Sketch the graphs of $y = 2f(x)$, $y = f(x + 3)$, and $y = -f(x) + 1$.

11. Let $f(x)$ be the function whose graph is on page 122 (#1 in the chapter 2 review exercises).

a) Find all x for which the function is discontinuous at x .

$$x = -3, 0, 2, 5$$

b) Find all a for which $\lim_{x \rightarrow a^+} f(x)$ does not exist.

$$a = 0$$

c) Find all a for which $\lim_{x \rightarrow a^-} f(x)$ does not exist.

$$a = 0, 2$$

d) Find all a for which $\lim_{x \rightarrow a} f(x)$ does not exist.

$$a = -3, 0, 2$$

12. Find $\lim_{x \rightarrow 0} \cos(x^2 - 2x)$, justifying your steps.

$$\lim_{x \rightarrow 0} \cos(x^2 - 2x) = \cos(\lim_{x \rightarrow 0} (x^2 - 2x)) = \cos(\lim_{x \rightarrow 0} (x^2) - \lim_{x \rightarrow 0} (2x)) = \cos(0 - 0) = 1$$

where the first equality follows from the fact that cosine is continuous, the second equality uses the laws of limits (since the limit of each piece exists), and the third equality comes by substituting in to the polynomials x^2 and $2x$.

13. What, if anything, is wrong with the following:

$$\lim_{x \rightarrow 0} \left(\frac{x}{\sin x} \right) = \frac{\lim_{x \rightarrow 0} (x)}{\lim_{x \rightarrow 0} (\sin x)} = \frac{0}{\lim_{x \rightarrow 0} (\sin x)} = 0.$$

The first equality is not justified by the limit laws because the limit in the denominator is zero.

14. Find the slope of the line tangent to the graph of $y = x^2 - 3x$ where $x = 4$. Use both formulas.

First formula:

$$\lim_{x \rightarrow 4} \frac{(x^2 - 3x) - (4^2 - 3(4))}{x - 4} = \lim_{x \rightarrow 4} \frac{x^2 - 3x - 4}{x - 4} = \lim_{x \rightarrow 4} \frac{(x - 4)(x + 1)}{x - 4} = \lim_{x \rightarrow 4} (x + 1) = 5$$

Second formula:

$$\lim_{h \rightarrow 0} \frac{[(4 + h)^2 - 3(4 + h)] - [4^2 - 3(4)]}{h} = \lim_{h \rightarrow 0} \frac{16 + 8h + h^2 - 12 - 3h - 16 + 12}{h} =$$
$$\lim_{h \rightarrow 0} \frac{h(5 + h)}{h} = 5$$