

1. Let $f(x)$ be the function whose graph is on the board.

a) Find all a for which the function is discontinuous at $x = a$.

$$a = -3, 0, 3, 5$$

b) Find all a for which $\lim_{x \rightarrow a^+} f(x)$ does not exist.

$$a = 0, 5$$

c) Find all a for which $\lim_{x \rightarrow a^-} f(x)$ does not exist.

$$a = 0$$

d) Find all a for which $\lim_{x \rightarrow a} f(x)$ does not exist.

$$a = -3, 0, 5$$

2. Find $\lim_{x \rightarrow 0} [\cos(x \sin(1/x))]$. (Hint: Use the fact that cosine is continuous and then apply the squeeze theorem to what's inside.)

Because cosine is continuous, we may write

$$\lim_{x \rightarrow 0} [\cos(x \sin(1/x))] = \cos[\lim_{x \rightarrow 0} (x \sin(1/x))].$$

Now we note that

$$-1 \leq \sin(1/x) \leq 1,$$

and so we may multiply through the inequality by x to obtain

$$-x \leq x \sin(1/x) \leq x.$$

But now we note that

$$\lim_{x \rightarrow 0} -x = 0 = \lim_{x \rightarrow 0} x,$$

and so the squeeze theorem tells us that

$$\lim_{x \rightarrow 0} x \sin(1/x) = 0.$$

Thus

$$\lim_{x \rightarrow 0} [\cos(x \sin(1/x))] = \cos[\lim_{x \rightarrow 0} (x \sin(1/x))] = \cos(0) = 1.$$

3. Write the precise ϵ - δ definition of the following: $\lim_{x \rightarrow a} f(x) = L$.

For each $\epsilon > 0$ there is some $\delta > 0$ so that $|f(x) - L| < \epsilon$ whenever $|x - a| < \delta$.

4. In this problem you will illustrate the definition of $\lim_{x \rightarrow 2} (3x - 5) = 1$.

(a) Find a value of δ that corresponds to $\epsilon = 1/4$. Be sure to show how you found δ .

We need to find a δ so that we have

$$|(3x - 5) - 1| < 1/4 \quad \text{whenever} \quad |x - 2| < \delta.$$

But the inequality on the left is equivalent to $|3x - 6| < 1/4$, or $3|x - 2| < 1/4$, or $|x - 2| < 1/12$. So we now find ourselves needing to find a δ so that we have

$$|x - 2| < 1/12 \quad \text{whenever} \quad |x - 2| < \delta.$$

Making $\delta = 1/12$ will do nicely.

(b) Show that this choice of δ does in fact work for this ϵ (i.e., show that when $\epsilon = 1/4$, the condition in the definition of limit is satisfied when δ is the one you've found).

We need to show that when $|x - 2| < 1/12$, we will have $|(3x - 5) - 1| < 1/4$. The argument is the same as above but backwards. $|x - 2| < 1/12$ is equivalent to $3|x - 2| < 1/4$, which is the same as $|3x - 6| < 1/4$, or, in other words, $|(3x - 5) - 1| < 1/4$, as required.

5. (a) Find the slope of the line tangent to the graph of $\sqrt{x^2 - 7}$ where $x = 4$ using the formula $\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$.

First note that $f(4) = 3$, so we are looking at the point $(4, 3)$. The slope of the tangent line here is given by

$$\lim_{x \rightarrow 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \rightarrow 4} \frac{\sqrt{x^2 - 7} - 3}{x - 4}.$$

We now multiply top and bottom by $\sqrt{x^2 - 7} + 3$, which is the conjugate of the numerator, and we obtain

$$\lim_{x \rightarrow 4} \left(\frac{\sqrt{x^2 - 7} - 3}{x - 4} \right) \left(\frac{\sqrt{x^2 - 7} + 3}{\sqrt{x^2 - 7} + 3} \right) = \lim_{x \rightarrow 4} \frac{x^2 - 7 - 9}{(x - 4)(\sqrt{x^2 - 7} + 3)}.$$

The numerator is now $x^2 - 16$, which factors as $(x + 4)(x - 4)$, and so we then cancel the two $x - 4$ terms to get

$$\lim_{x \rightarrow 4} \frac{x + 4}{\sqrt{x^2 - 7} + 3}.$$

This is defined when $x = 4$, so we may plug in to obtain

$$8/6 = 4/3.$$

(b) What is the other formula for the slope of the tangent line to $y = f(x)$ at $x = a$?

The other formula is

$$\lim_{h \rightarrow 0} \frac{f(a + h) - f(a)}{h}.$$

6. Complete the following statement of the intermediate value theorem:

Suppose that f is continuous on the closed interval $[a, b]$ and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then...

there is some number c with $a < c < b$ so that $f(c) = N$.

7. Use the intermediate value theorem to show that there is some number equal to its cosine.

We are looking for a solution to the equation $\cos(x) = x$. In other words, we are looking for an x so that $\cos(x) - x = 0$. Call this function $f(x)$, so we have

$$f(x) = \cos(x) - x.$$

Note that

$$f(0) = \cos(0) - 0 = 1 - 0 = 1 > 0,$$

while

$$f(\pi/2) = \cos(\pi/2) - \pi/2 = 0 - \pi/2 = -\pi/2 < 0.$$

Thus by the intermediate value theorem, there is some number c , between 0 and $\pi/2$ so that $f(c) = 0$. In other words, $\cos(c) - c = 0$, or $\cos(c) = c$.