Calculus I **Test 2 Solutions** **October 25, 2004**

1. Find the limit $\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)}$.

$$\lim_{x\to 0} \frac{\sin(3x)}{\sin(2x)} = \lim_{x\to 0} \frac{3x}{2x} \cdot \frac{\sin(3x)}{3x} \cdot \frac{2x}{\sin(2x)} =$$

$$\frac{3}{2}\left(\lim_{3x\to 0} \frac{\sin(3x)}{3x}\right)\left(\lim_{x\to 0} \frac{2x}{\sin(2x)}\right) = \frac{3}{2}$$

2. Use the derivatives of $\sin(x)$ and $\cos(x)$ to show directly that $\frac{d}{dx}\tan(x) = \sec^2(x)$ (recall that $\tan(x) = \frac{\sin(x)}{\cos(x)}$).

Quotient rule:

$$\frac{d}{dx}\tan(x) = \frac{d}{dx}\frac{\sin(x)}{\cos(x)} = \frac{(\sin(x))'\cos(x) - \sin(x)(\cos(x))'}{\cos^2(x)} =$$

$$\frac{\cos(x)\cos(x) - (-\sin(x))\sin(x)}{\cos^2(x)} = \frac{\cos^2(x) + \sin^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)} = \sec^2(x)$$

3. (a) Verify that the following families of curves are orthogonal trajectories of one another:

$$y = a\sqrt{x} \qquad 2x^2 + y^2 = b$$

Implicit differentiation:
For $y = a\sqrt{x}$, we obtain $y' = \frac{a}{2\sqrt{x}}$. Note that from $y = a\sqrt{x}$ we have that $a = y/\sqrt{x}$, so we get

$$y' = \frac{y}{2x}.$$

For $2x^2 + y^2 = b$ we obtain $4x + 2yy' = 0$, so

$$y' = \frac{-2x}{y}.$$

These are negative reciprocals of one another, so the families are orthogonal.

(b) Find the equation for the line tangent to the graph of $2x^2 + y^2 = 36$ at the point $(4,1)$ (some of your work from part (a) might be helpful).

NOTE: The point should have been $(4,2)$, as $(4,1)$ isn't on the graph. This error was inconsequential to the problem, however.

From part (a), we know that $y' = -2x/y$ for this graph. Thus at $(4,1)$, we have a slope of $-2(4)/1 = -8$. The line with this slope through the point $(4,1)$ is $y - 1 = -8(x-4)$, or

$$y = -8x + 33.$$

(c) At what points does this curve $2x^2 + y^2 = 36$ have a horizontal tangent?

Again, we know from part (a) that $y' = -2x/y$ for this graph. Thus $y' = 0$ whenever $x = 0$. When $x = 0$ we find (using $2x^2 + y^2 = 36$) that $y = \pm 6$. So the two points are

$$(0, 6), (0, -6).$$

4. Answer one of the following two questions. For either version, you should explain you answer. Your answer should also include a complete expression for $r'(x)$ or $\frac{dr}{dx}$. (In fact, that's a really good place to start.)

Version I: Suppose $r(x) = f(g(h(x)))$, where $h'(1) = 0$. What is $r'(1)$?

Version II: Suppose $r = f(u)$, $u = g(w)$, and $w = h(x)$, where $\frac{dw}{dx}(1) = 0$. What is $\frac{dr}{dx}$ where $x = 1$?

VERSION I: Note that $r'(1) = f'(g(h(x))) \cdot g'(h(x)) \cdot h'(x)$. So when $x = 1$ we get

$$r'(1) = f'(g(h(1))) \cdot g'(h(1)) \cdot h'(1).$$

Because $h'(1) = 0$, this entire product must be zero, so

$$r'(1) = 0.$$

VERSION II: Note that $\frac{dr}{dx} = \frac{dr}{du}\frac{du}{dw}\frac{dw}{dx}$. When $x = 1$, the term $\frac{dw}{dx}$ is zero, so the entire product is zero. Thus

$$\frac{dr}{dx} = 0 \text{ when } x = 1.$$

5. A particle is moving up and down with its vertical displacement at time t given by $s(t) = (t^2 - 3)^{5/2}$.

(a) Find an equation for the velocity of the particle at time t.

Chain rule:

$$v(t) = s'(t) = \frac{5}{2}(t^2 - 3)^{3/2}(2t) = 5t(t^2 - 3)^{3/2}.$$

(b) Find the acceleration at time $t = 2$.

Product rule and chain rule:

$$a(t) = v'(t) = s''(t) = 5(t^2 - 3)^{3/2} + \frac{3}{2}(t^2 - 3)^{1/2}(2t)(5t) = 5(t^2 - 3)^{3/2} + 15t(t^2 - 3)^{1/2},$$

$$a(2) = 5 + 15(2) = 35.$$

6. Find $\frac{dy}{dx}$, where $y = \sin^3(2x)$

Chain rule twice: This is a composition of three functions. Either $y = f(g(h(x)))$, where $f(x) = x^3$, $g(x) = \sin(x)$, and $h(x) = 2x$, or $y = u^3$, $u = \sin(w)$, and $w = 2x$. Either way we get

$$\frac{dy}{dx} = 3(\sin(2x))^2 \cdot (\cos(2x)) \cdot (2) = 6 \sin^2(2x) \cos(2x).$$