Calculus I **Test 3 Solutions** **December 3, 2004**

1. Use a tangent line approximation to estimate the value of $\sqrt{10}$.

Let $f(x) = \sqrt{x}$. We will use the tangent line to $y = f(x)$ at $x = 9$ to extimate $f(10) = \sqrt{10}$. To find the equation for the tangent line, note that $f(9) = 3$, and $f'(9) = 1/2\sqrt{9} = 1/6$. So the tangent line has equation

$$y - 3 = \frac{1}{6}(x - 9) \qquad \text{or} \qquad y = \frac{1}{6}x + \frac{3}{2}.$$

So we have

$$\sqrt{10} = f(10) \approx \frac{10}{6} + \frac{3}{2} = \frac{19}{6}.$$

2. (a) Find the horizontal and vertical asymptotes of the function $f(x) = \dfrac{\sqrt{9x^4+1}}{x^2-4}$.

Vertical asymptotes are where the denominator equals zero. This happens when $x = \pm 2$. To find horizontal asymptotes, we calculate

$$\lim_{x\to\pm\infty} \frac{\sqrt{9x^4+1}}{x^2-4} = \lim_{x\to\pm\infty} \frac{\sqrt{9+\frac{1}{x^2}}}{1-\frac{1}{x^2}} = 9.$$

So the line $y = 9$ is the horizontal asymptote.

For parts (b)-(e), use the function $f(x) = x^4 - 6x^2 + 5$.

(b) Find all critical numbers and intercepts of $f(x)$.

The y-intercept is $f(0) = 5$. For the x-intercepts, we have $x^4 - 6x^2 + 5 = (x^2 - 5)(x^2 - 1)$, which is zero when $x = \pm\sqrt{5}$ or $x = \pm 1$. For the critical numbers, we have $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$, which is zero when $x = 0$ or $x = \pm\sqrt{3}$.

(c) For each of the critical numbers found in part (b), use the second derivative test to determine whether the function has a local maximum, local minimum, or neither at that point.

The second derivative is $f''(x) = 12x^2 - 12$, so $f''(0) = -12$, meaning that a maximum occurs when $x = 0$. Also $f''(\sqrt{3}) = f''(-\sqrt{3}) = 36 - 24 = 12$, meaning that a minimum occurs when $x = \pm\sqrt{3}$.

(d) Determine the inflection points of $f(x)$.

To find the inflection points, note that $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$, so inflection points occur when $x = \pm 1$.

(e) Sketch the graph of $y = f(x)$. Be sure to label all of the points found in parts (b)-(d).

It has a local maximum at $(0, 5)$, from which it descends, concave down, to the x-axis on both sides to the points $(\pm 1, 0)$. It then becomes concave up and proceeds to the points $(\pm\sqrt{3}, -4)$, at which there are local minima. It then comes back to the x-axis, hitting it at the points $(\pm\sqrt{5}, 0)$, and continues up in either direction indefinitely.

3. (a) State the Mean Value Theorem.

If f is continuous on $[a, b]$ and differentiable on (a, b), then there is a number c between a and b so that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

(b) Use Rolle's Theorem to show that $f(x) = x^4 + 4x - 1$ has at most two roots. Be sure to explain your logic.

Note that $f'(x) = 4x^3 + 4 = 4(x^3 + 1)$, which has only one root, $x = 0$. Thus f has at most two roots.

(c) Suppose f is a continuous differentiable function so that $f(0) = 3$ and $f'(x) \leq 2$ for all x. What is the largest possible value for $f(4)$? Explain.

Using the mean value theorem, we have that for some c between 0 and 4

$$2 \geq f'(c) = \frac{f(4) - f(0)}{4 - 0}.$$

Algebra shows then that

$$f(4) \leq 11.$$

4. You are standing on the shore of a lake holding one end of a rope, the other end of which is tied to a boat on the lake. You want to pull the boat to shore. In order to make use of your weight, you throw the rope over a tree limb 14 ft high. You then are able to pull the rope straight down at a rate of $4/5$ (or 0.8) feet per second.

(a) Draw a picture showing all relevant information, labeling the distance of the boat from the shore as D and the length of rope between the tree and the boat as L. What quantity is changing at $4/5$ feet per second? (Write the answer as a derivative.)

This is a right triangle with hypotenuse labeled L, base labeled D, and height labeled 14. Note that although you are pulling down on the rope, the limb is not moving, so the 14 is always 14. Pulling the rope makes L decrease at a rate of $4/5$ feet per second, so

$$\frac{dL}{dt} = -\frac{4}{5}.$$

(b) Express L as a function of D.

Using the Pythagorean theorem, we have that $L^2 = D^2 + 14^2$. Because L is positive, we can solve for L to obtain

$$L = \sqrt{D^2 + 196}.$$

(c) Use your function from part (b) to determine how fast the boat is moving when the boat is 48 feet away from shore.

Taking derivatives, we find that

$$\frac{dL}{dt} = \frac{D}{\sqrt{D^2 + 196}} \frac{dD}{dt}.$$

Plugging in $D = 48$ and $dL/dt = -4/5$, and using the fact that $48^2 + 14^2 = 2500 = 50^2$, we obtain

$$\frac{dD}{dt} = -\frac{4}{5}\frac{50}{48} = -\frac{5}{6}.$$

5. In order to deliver a package, the postal service requires that the sum of the length and the girth be no more than 130 inches ("length" always refers to the longest side; "girth" is how far it is around the middle — see the picture on the board). To avoid extra postage, they also require that the length be no more than 34 inches. Use calculus to find the maximum volume that a box meeting these requirements can hold.

To maximize volume, we set the length to its maximum of 34. Note that girth is $2w + 2h$, so maximizing the other dimensions requires that $\ell + 2w + 2h = 130$, or $w+h = 48$. The formula for volume is $V = \ell wh$. Plugging in $\ell = 34$ and $h = 48-w$, we obtain

$$V(w) = 34w(48 - w) = 34(48w - w^2).$$

Taking derivatives we have

$$V'(w) = 34(48 - 2w).$$

This is zero when $w = 24$. When w is 24, we have that $h = 24$ and $l = 34$, so the maximum volume is

$$(34)(24)(24) = 19,584.$$