Introduction to Bases

A collection of vectors $S = \{\mathbf{v_1}, \ldots, \mathbf{v_n}\}$ is said to form a basis for a vector space V if S spans $V$and S is linearly independent. Bases are not at all unique, but we do have some favorites. For example, the standard basis for $\mathbf{R}^3$ consists of the vectors $\mathbf{e_1} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{e_2} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{e_3} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The standard basis for P_3 is $\{t^3, t^2, t, 1\}$.

A basis should be thought of as a collection of vectors complete enough to span the whole space yet with no more vectors than necessary to do so. This way of thinking is justified by the following facts:

1. Any set S which spans V contains a basis for V as a subset.
2. Any set S of linearly independent vectors in V can be extended to a basis for V.

Finally, any two bases for a given vector space V will have the same number of vectors. This number (if it is finite) is called the dimension of V. From this fact we deduce the following:

How-to

To check if a set S forms a basis for a vector space V, one needs to check that S spans V and that S is linearly independent, as explained in the "Spans" and "Linear Independence" handouts.

If V is known to be an n-dimensional vector space and S consists of n vectors, then one need only check one of the two conditions, either that S spans V or that S is linearly independent.

Given a set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of vectors spanning $\mathbf{R}^n$ (so $k \geq n$), one can find a basis in S by forming a matrix whose ith column is the vector $\mathbf{v_i}$ and then row reducing this matrix. The vectors corresponding to the columns in which the initial 1s appear form a basis. For instance, if the initial ones appear in the first, third and fourth columns, then the first, third and fourth vectors of S form a basis for V.

Given a set $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of vectors which are linearly independent in $\mathbf{R}^n$ (so $n \leq k$), one can extend these to a basis for $\mathbf{R}^n$ by forming a matrix whose first k columns are the vectors of S, and then adding n more columns consisting of the standard basis vectors $\mathbf{e_1}, \ldots, \mathbf{e_n}$. As before, one then row reduces this matrix, and the vectors corresponding to the columns containing the initial 1s will form a basis. The first k columns will always contain initial 1s, so the basis you form will contain S.