Practice problems for Linear Algebra Test II.

1. (a) Find a basis S for $\mathbf{R}^3$ that includes the vectors $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ -2 \end{bmatrix}$.

(b) Use the basis S you found in part (a) to find a basis for $\mathbf{R}^3$ which is orthonormal with respect to the standard dot product on $\mathbf{R}^3$. (Hint: First find an orthogonal basis, clearing fractions as you go; at the end you can divide each vector by its length.)

(c) Suppose $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ is a basis for $\mathbf{R}^3$, and suppose the transition matrix from T to S is $P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. Find T.

(d) Use the matrix from part (c) to express $\mathbf{v} = \mathbf{w}_1 + 2\mathbf{w}_2 - \mathbf{w}_3$ in terms of the basis S from part (a).

(e) Check your answer in (d) by showing that the expressions of $\mathbf{v}$ in terms of T and of S give the same vector.

2. (a) Given the basis $S = \{t+1, t-1\}$ for P_1, find basis for P_1 which is orthonormal with respect to the inner product $(f, g) = \int_0^1 f(t)g(t)\,dt$.

(b) Suppose $T = \{\mathbf{w}_1, \mathbf{w}_2\}$ is a basis for P_1, and suppose the transition matrix from T to S is $P_{S \leftarrow T} = \begin{bmatrix} 3 & -2 \\ 2 & 1 \end{bmatrix}$. Find T.

(c) Use the matrix from part (c) to express $\mathbf{v} = -\mathbf{w}_1 + 2\mathbf{w}_2$ in terms of the basis S from part (a).

(d) Check your answer in (c) by showing that the expressions of $\mathbf{v}$ in terms of T and of S give the same vector.

3. Let W be the subspace of M_{22} of symmetric 2×2 matrices. Find the dimension of W by exhibiting an isomorphism $f : W \to \mathbf{R}^n$ for some n. Verify that the function you describe is in fact an isomorphism (you do not have to check that it is one-to-one and onto, however, just that it preserves the vector space operations, properties (a) and (b) in the definition on page 190). (Hint: Recall that *symmetric* means that $A = A^T$. What does this mean the entries look like?)

4. (a) Find the length of the vector $3t - 2$ in P_1 where the inner product is defined as $(at + b, ct + d) = ac - 2ad - 2bc + 5bd$.

(b) Find the angle between the vectors $\cos(t)$ and $\sin(t)$ where V is the space of continuous functions with inner product $(f, g) = \int_0^1 f(t)g(t)\,dt$.

(c) State the Cauchy-Schwarz inequality for an arbitrary inner product space V with inner product $(\, , \,)$.

5. Find the Fourier polynomial of degree two for the function $f(t) = \sin^2 t$ using the following facts:

$$\int_{-\pi}^{\pi} \sin^2 t \, dt = \pi \qquad \int_{-\pi}^{\pi} \sin^2 t \cos t \, dt = 0 \qquad \int_{-\pi}^{\pi} \sin^3 t \, dt = 0$$

$$\int_{-\pi}^{\pi} \sin^2 t \cos 2t \, dt = -\pi/2 \qquad \int_{-\pi}^{\pi} \sin^2 t \sin 2t \, dt = 0$$