Solutions to practice problems for Linear Algebra Test III.

1. Suppose $L: V \to W$ is a linear transformation, where the dimension of V is n and the dimension of W is m.

(a) State the rank-nullity theorem for L.

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$$

(b) Suppose $n = 7$ and $m = 4$. If L is onto, what is the dimension of the kernel of L? Explain.

Ans: 3

(c) Suppose $V = W$ and L is one-to-one. What else can you say about L? Explain.

Ans: L is also onto (and so is an isomorphism).

2. Verify the rank-nullity theorem for $L: \mathbf{R}^3 \to \mathbf{R}^2$, where

$$L\left(\begin{bmatrix} a \\ b \\ c \end{bmatrix}\right) = \begin{bmatrix} a - b + c \\ -a + b - c \end{bmatrix},$$

Ans: The kernel is two dimensional and the range is one dimensional. As the domain $\mathbf{R}^3$ is three dimensional, we have verified the rank-nullity theorem, as $2 + 1 = 3$.

3. Use determinants and cofactors to find the inverse of the following matrix:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

Ans:

$$\begin{bmatrix} -4 & 1 & -2 \\ -7 & -2 & 4 \\ -2 & -7 & -1 \end{bmatrix}.$$

4. Suppose $L: P_2 \to \mathbf{R}^2$ is defined by

$$L(p(t)) = \begin{bmatrix} p'(1) \\ \int_0^1 p(t)\,dt \end{bmatrix}.$$

(a) Verify that L is linear.

(b) Let $S = \{t^2 + 1, t + 1, t - 1\}$ be a basis for P_2, and $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ be a basis for $\mathbf{R}^2$. Find the matrix representation for L with respect to S and T.

Ans:

$$\begin{bmatrix} 5/3 & 5/4 & 1/4 \\ 1/3 & -1/4 & 3/4 \end{bmatrix}.$$

5. Suppose $L: \mathbf{R}^2 \to \mathbf{R}^2$ is defined by

$$L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a - 3b \\ 2a + b \end{bmatrix}.$$

(a) Suppose $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is the standard basis for $\mathbf{R}^2$. Find the matrix representation for L with respect to S.

Ans:

$$\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}.$$

(b) Use transition matrices to find the matrix representation for L with respect to the basis $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$.

Ans:

$$-\frac{1}{2} \begin{bmatrix} -1 & -5 \\ 5 & -3 \end{bmatrix}.$$

6. Find the determinants of the following matrices:

(a) $\begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ (c) $\begin{bmatrix} 1 & 7 & 4 & -3 \\ 2 & -9 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 3 & 5 & 2 & 9 & 0 \\ 5 & 3 & 9 & 5 & 1 \\ 0 & 1 & 4 & 2 & 9 \\ 1 & 8 & 5 & 2 & 0 \\ 5 & 3 & 9 & 5 & 1 \end{bmatrix}$

Ans:

(a) -10

(b) 5

(c) 22

(d) 0