

Solutions to practice problems for Linear Algebra Test III.

1. Suppose  $L: V \rightarrow W$  is a linear transformation, where the dimension of  $V$  is  $n$  and the dimension of  $W$  is  $m$ .

(a) State the rank-nullity theorem for  $L$ .

$$\dim(\ker(L)) + \dim(\text{range}(L)) = \dim(V)$$

(b) Suppose  $n = 7$  and  $m = 4$ . If  $L$  is onto, what is the dimension of the kernel of  $L$ ? Explain.

Because  $L$  is onto, we know that the dimension of the image is the same as the dimension of  $W$ , which is 4. Thus the rank-nullity equation becomes  $\dim(\ker(L)) + 4 = 7$ , so the dimension of the kernel is 3.

(c) Suppose  $V = W$  and  $L$  is one-to-one. What else can you say about  $L$ ? Explain.

Because  $V = W$ , we know  $n = m$ . Also,  $L$  being one-to-one means that the kernel has dimension zero. Thus the rank-nullity equation becomes  $0 + \dim(\text{range}(L)) = n$ , so the dimension of the range is  $n$ . But because this is exactly the dimension of  $W$ , we deduce that  $L$  is onto. Because  $L$  is one-to-one and onto, this means  $L$  is an isomorphism.

2. Verify the rank-nullity theorem for  $L: \mathbf{R}^3 \rightarrow \mathbf{R}^2$ , where

$$L \left( \begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a - b + c \\ -a + b - c \end{bmatrix},$$

To find the kernel, we set  $L(\mathbf{v}) = 0$  and solve for  $\mathbf{v}$ . So we get  $a - b + c = 0$  and  $-a + b - c = 0$ . From these we get  $c = b - a$ , so the kernel is the set of all vectors of the form

$$\begin{bmatrix} a \\ b \\ b - a \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

It follows that

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

spans the kernel. Because these vectors are linearly independent, they form a basis. In particular, the kernel is two dimensional.

The range consists of all vectors of the form

$$\begin{bmatrix} a - b + c \\ -a + b - c \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} -1 \\ 1 \end{bmatrix} + c \begin{bmatrix} 1 \\ -1 \end{bmatrix}.$$

These three vectors thus span the range, and so contain a basis. As they are all multiples of the same vector  $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ , we deduce that the range is one dimensional

(alternatively, you can put the vectors all together in a matrix and row reduce; the basis corresponds to the columns with the initial ones).

As the domain  $\mathbf{R}^3$  is three dimensional, we have verified the rank-nullity theorem, as  $2 + 1 = 3$ .

3. Use determinants and cofactors to find the inverse of the following matrix:

$$\begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & -2 \\ 3 & 2 & 1 \end{bmatrix}$$

The various cofactors are

$$\begin{aligned} A_{11} &= (0)(1) - (-2)(-2) = 4 & A_{12} &= (1)(1) - (3)(-2) = 7 & A_{13} &= (1)(2) - (3)(0) = 2 \\ A_{21} &= (-1)(1) - (-2)(0) = -1 & A_{22} &= (2)(1) - (3)(0) = 2 & A_{23} &= (2)(2) - (3)(-1) = 7 \\ A_{31} &= (-1)(-2) - (0)(0) = 2 & A_{32} &= (2)(-2) - (1)(0) = -4 & A_{33} &= (2)(0) - (1)(-1) = 1. \end{aligned}$$

Using the first three of these, we find that the determinant of the matrix is  $4 - 7 + 2 = -1$ . The inverse is obtained by putting the cofactors in their transpose positions and dividing everything by the determinant. We thus obtain

$$\begin{bmatrix} -4 & 1 & -2 \\ -7 & -2 & 4 \\ -2 & -7 & -1 \end{bmatrix}.$$

4. Suppose  $L: P_2 \rightarrow \mathbf{R}^2$  is defined by

$$L(p(t)) = \begin{bmatrix} p'(1) \\ \int_0^1 p(t) dt \end{bmatrix}.$$

(a) Verify that  $L$  is linear.

Let  $p(t)$  and  $q(t)$  be two vectors in  $P_2$ . Then

$$L(p(t) + q(t)) = \begin{bmatrix} (p(t) + q(t))' \\ \int_0^1 p(t) + q(t) dt \end{bmatrix} = \begin{bmatrix} p'(t) + q'(t) \\ \int_0^1 p(t) dt + \int_0^1 q(t) dt \end{bmatrix} = L(p(t)) + L(q(t))$$

and for any real number  $r$  we have

$$L(rp(t)) = \begin{bmatrix} (rp(t))' \\ \int_0^1 rp(t) dt \end{bmatrix} = \begin{bmatrix} rp'(t) \\ r \int_0^1 p(t) dt \end{bmatrix} = rL(p(t)).$$

(b) Let  $S = \{t^2 + 1, t + 1, t - 1\}$  be a basis for  $P_2$ , and  $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$  be a basis for  $\mathbf{R}^2$ . Find the matrix representation for  $L$  with respect to  $S$  and  $T$ .

Applying  $L$  to the vectors of  $S$  we obtain

$$L(t^2 + 1) = \begin{bmatrix} 2 \\ 4/3 \end{bmatrix}, \quad L(t + 1) = \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}, \quad L(t - 1) = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}.$$

We now set each of these vectors equal to  $a \begin{bmatrix} 1 \\ 1 \end{bmatrix} + b \begin{bmatrix} 1 \\ -1 \end{bmatrix}$  and solve for  $a$  and  $b$  to obtain the coresponding  $T$ -coordinates. We get

$$\begin{bmatrix} 2 \\ 4/3 \end{bmatrix}_T = \begin{bmatrix} 5/3 \\ 1/3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 3/2 \end{bmatrix}_T = \begin{bmatrix} 5/4 \\ -1/4 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -1/2 \end{bmatrix}_T = \begin{bmatrix} 1/4 \\ 3/4 \end{bmatrix}.$$

It follows that the matrix we seek is

$$\begin{bmatrix} 5/3 & 5/4 & 1/4 \\ 1/3 & -1/4 & 3/4 \end{bmatrix}.$$

5. Suppose  $L: \mathbf{R}^2 \rightarrow \mathbf{R}^2$  is defined by

$$L \left( \begin{bmatrix} a \\ b \end{bmatrix} \right) = \begin{bmatrix} a - 3b \\ 2a + b \end{bmatrix}.$$

(a) Suppose  $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$  is the standard basis for  $\mathbf{R}^2$ . Find the matrix representation for  $L$  with respect to  $S$ .

As above, we find that

$$L \left( \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad L \left( \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -3 \\ 1 \end{bmatrix}.$$

As  $S$  is the standard basis, the matrix we seek is

$$\begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix}.$$

(b) Use transition matrices to find the matrix representation for  $L$  with respect to the basis  $T = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ .

The transition matrix from  $T$  to  $S$  is

$$P = P_{S \leftarrow T} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(Recall that the transition matrix from  $T$  to  $S$  has columns equal to the  $S$ -coordinates of the vectors of  $T$ .) The transition matrix from  $S$  to  $T$  is

$$P_{T \leftarrow S} = P^{-1} = -\frac{1}{2} \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

It follows that the matrix we seek is

$$P^{-1} \begin{bmatrix} 1 & -3 \\ 2 & 1 \end{bmatrix} P = -\frac{1}{2} \begin{bmatrix} -1 & -5 \\ 5 & -3 \end{bmatrix}.$$

6. Find the determinants of the following matrices:

$$(a) \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} \quad (b) \begin{bmatrix} 1 & 2 & -1 \\ 0 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad (c) \begin{bmatrix} 1 & 7 & 4 & -3 \\ 2 & -9 & 0 & 1 \\ 0 & 2 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad (d) \begin{bmatrix} 3 & 5 & 2 & 9 & 0 \\ 5 & 3 & 9 & 5 & 1 \\ 0 & 1 & 4 & 2 & 9 \\ 1 & 8 & 5 & 2 & 0 \\ 5 & 3 & 9 & 5 & 1 \end{bmatrix}$$

(a)

$$(2)(1) - (4)(3) = -10$$

(b) Expanding along the top row we have

$$1\left((2)(1) - (1)(1)\right) - 2\left((0)(1) - (1)(1)\right) - 1\left((0)(1) - (1)(2)\right) = 1 + 2 + 2 = 5$$

(c) Expanding along the third row we have

$$-2\det \begin{bmatrix} 1 & 4 & -3 \\ 2 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix} = -2\left(1(0 - 1) - 4(2 - 1) - 3(2 - 0)\right) = -2(-1 - 4 - 6) = 22$$

(d) The second and fifth rows are equal, so the determinant is zero.