Introduction to Spans

The span of a set of vectors is the collection of all possible linear combinations of those vectors. A collection of vectors is said to span a space if that space is the span of the vectors. In other words, a collection of vectors $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ spans a space V if every vector $\mathbf{v}$ in V can be written as $\mathbf{v} = a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k}$ for some real numbers $a_1, \ldots, a_k$.

How-to

1. To determine if a particular vector $\mathbf{v}$ is in the span of a particular set of vectors $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$, you write $a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{v}$ and then try to solve for $a_1, \ldots, a_k$. If you find a solution, then you have found how to write $\mathbf{v}$ as a linear combination of the vectors in S, and so $\mathbf{v}$ is indeed in span(S). If there is no solution, then $\mathbf{v}$ cannot be written as a linear combination of the vectors in S, and so $\mathbf{v}$ is not in span(S).

Note: How you actually determine if there are such $a_1, \ldots, a_k$ depends on the context; i.e., it depends on the vector space V you are actually dealing with. In the examples we will do, it will always come down to solving a system of linear equations.

For example, if $V = \mathbf{R}^n$, then the equation $a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{v}$ has an $n \times 1$ vector on either side, and so turns into a system of n linear equations obtained by considering the entries of $\mathbf{v}$ one at a time.

If $V = P_n$, then the equation $a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{v}$ has an n degree polynomial on either side, and so turns into a system of $n + 1$ linear equation obtained by equating the coefficients.

2. To determine whether a given collection $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of vectors in a vector space V actually spans all of V, you set up the equation $a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{v}$ but now with $\mathbf{v}$ unspecified. Again you try to solve for $a_1, \ldots, a_k$, but because you haven't specified $\mathbf{v}$, your solutions will depend on $\mathbf{v}$.

3. To determine whether a given collection $S = \{\mathbf{v_1}, \ldots, \mathbf{v_k}\}$ of vectors in a vector space V spans a subspace W of V, you again set up the equation $a_1\mathbf{v_1} + \cdots + a_k\mathbf{v_k} = \mathbf{v}$ but now with $\mathbf{v}$ an unspecified vector in W, and then proceed as above.