Linear Algebra I **Test 1 Solutions** **September 24, 2004**

1. Let $A = \begin{bmatrix} -3 & 1 \\ 1 & 0 \\ 2 & 7 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 2 & -1 & 4 \\ 5 & -4 & -3 & 3 \end{bmatrix}$, and $C = \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 5 & 4 \end{bmatrix}$.

If possible, find (a) B^T (b) AC (c) AB

$$B^T = \begin{bmatrix} 0 & 5 \\ 2 & -4 \\ -1 & -3 \\ 4 & 3 \end{bmatrix}$$

AC is not defined

$$AB = \begin{bmatrix} 5 & -10 & 0 & -9 \\ 0 & 2 & -1 & 4 \\ 35 & -24 & -23 & 29 \end{bmatrix}$$

2. Suppose $C = \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & -1 \\ -2 & 5 & 4 \end{bmatrix}$.

(a) Use Gaussian-type elimination to find C^{-1}. In this computation, indicate the first point at which C is in row-echelon form and reduced row-echelon form.

$$C^{-1} = \begin{bmatrix} -5 & 7 & -2 \\ 2 & -2 & 1 \\ -5 & 6 & -2 \end{bmatrix}$$

(b) Use C^{-1} to solve the following system of linear equations:

$$2x_1 - 2x_2 - 3x_3 = 1$$

$$x_1 - x_3 = 1$$

$$-2x_1 + 5x_2 + 4x_3 = 1$$

The system is equivalent to $C\mathbf{x} = \mathbf{b}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. We can multiply each side of this equation *on the left* by C^{-1} to obtain $\mathbf{x} = C^{-1}\mathbf{b}$, and so

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.$$

3. Determine for which values of a the following system has 0 solutions, 1 solution, and infinitely many solutions.

$$x + 2y = 3$$

$$4x + (a^2 - 1)y = a + 15$$

The corresponding augmented matrix row reduces to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & a^2 - 9 & a + 3 \end{bmatrix}.$$

As long as $|a| \neq 3$, we can further reduce to

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & \frac{1}{a-3} \end{bmatrix},$$

which has the unique solution

$$x = 3 - \frac{2}{a-3} \qquad y = \frac{1}{a-3}.$$

To see what happens when $|a| = 3$, we note that for $a = 3$ the matrix becomes

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 6 \end{bmatrix},$$

whose corresponding linear system has no solution. On the other hand, when $a = -3$, the matrix becomes

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \end{bmatrix},$$

which has infintely many solutions.

4. Find all 2×2 matrices $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ with the property that A^2 is the zero matrix.

With A as above, note that

$$A^2 = \begin{bmatrix} a^2 & ab + bd \\ 0 & d^2 \end{bmatrix}.$$

Setting each of these entries to zero, we see that $a = 0 = d$, while b may be any real number. Thus the answer is all matrices A of the form

$$\begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix}.$$

5. Suppose V is the set of all positive diagonal 2×2 matrices (so V is the set of all matrices of the form $\begin{bmatrix} \alpha & 0 \\ 0 & \beta \end{bmatrix}$ where $\alpha, \beta > 0$). Define $\oplus$ to be ordinary matrix multiplication, and define $\odot$ to be exponentiation of the entries of A (i.e., if the matrix $\mathbf{u}$ has diagonal entries α, β, then $c \odot \mathbf{u}$ has diagonal entries α^c, β^c). Then V, with these operations, is a vector space.

(a) Property (3) of vector spaces states that there exists an element $\mathbf{0}$ in V such that $\mathbf{u} \oplus \mathbf{0} = \mathbf{u}$ for any $\mathbf{u}$ in V. What is $\mathbf{0}$ here?

Write $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ where $a, b > 0$. Note that because $\mathbf{0}$ is required to be in V, it will be of the form $\mathbf{0} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ for some $x, y > 0$. Thus we need to find x and y so that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$$

for all $a, b > 0$. So we need $ax = a$ and $by = b$, or $x = y = 1$. Thus we have

$$\mathbf{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

(b) Property (4) of vector spaces states that for each $\mathbf{u}$ in V there exists an element $-\mathbf{u}$ in V such that $\mathbf{u} \oplus -\mathbf{u} = \mathbf{0}$. Given an element $\mathbf{u}$ of V, what is $-\mathbf{u}$?

Again, let $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, where $a, b > 0$. Again, because $-\mathbf{u}$ is required to be in V, we will have $-\mathbf{u} = \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix}$ for some $x, y > 0$. Also, from above we have $\mathbf{0} = I_2$, so we end up needing to have $ax = 1$ and $by = 1$, or $x = 1/a$ and $y = 1/b$. Thus we have

$$-\mathbf{u} = \begin{bmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{bmatrix}.$$

(c) Verify directly that $\mathbf{u} \oplus \mathbf{v} = \mathbf{v} \oplus \mathbf{u}$ for all $\mathbf{u}$ and $\mathbf{v}$ in V.

Let $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix}$. Then

$$\mathbf{u} \oplus \mathbf{v} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \cdot \begin{bmatrix} s & 0 \\ 0 & t \end{bmatrix} = \begin{bmatrix} as & 0 \\ 0 & bt \end{bmatrix}.$$

It is easily checked that $\mathbf{v} \oplus \mathbf{u}$ gives the same matrix.

(d) Verify directly that $c \odot (d \odot \mathbf{u}) = (cd) \odot \mathbf{u}$ for all $\mathbf{u}$ in V and real numbers c and d.

With $\mathbf{u} = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, we have that

$$c \odot (d \odot \mathbf{u}) = c \odot \left(\begin{bmatrix} a^d & 0 \\ 0 & b^d \end{bmatrix} \right) = \begin{bmatrix} a^{dc} & 0 \\ 0 & b^{dc} \end{bmatrix} = \begin{bmatrix} a^{cd} & 0 \\ 0 & b^{cd} \end{bmatrix} = (cd) \odot \mathbf{u}.$$

6. (a) Verify that the 2×1 matrices of the form $\begin{bmatrix} x \\ 3x \end{bmatrix}$ form a subspace of $\mathbf{R}^2$.

Choose some $\mathbf{u}$ and $\mathbf{v}$ in V. Because they are in V, we may write them as $\mathbf{u} = \begin{bmatrix} a \\ 3a \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} b \\ 3b \end{bmatrix}$ for some a and b. We then need to check that $\mathbf{u} \oplus \mathbf{v}$ can also be written as $\begin{bmatrix} x \\ 3x \end{bmatrix}$ for some x. But

$$\mathbf{u} \oplus \mathbf{v} = \begin{bmatrix} a + b \\ 3a + 3b \end{bmatrix},$$

which is of the appropriate form (with $x = a + b$).

We now need to show that given any real number c we have that $c \odot \mathbf{u}$ is of the form $\begin{bmatrix} x \\ 3x \end{bmatrix}$ for some x. But

$$c \odot \mathbf{u} = \begin{bmatrix} ca \\ 3ca \end{bmatrix},$$

which is of the appropriate form (with $x = ca$).

Thus these matrices do form a subspace of $\mathbf{R}^2$.

(b) Are the 2×1 matrices of the form $\begin{bmatrix} x \\ x^2 \end{bmatrix}$ a subspace of $\mathbf{R}^2$? Justify your answer.

These matrices do not form a subspace of $\mathbf{R}^2$, as they fail both required properties. For example, if we let $\mathbf{u} = \begin{bmatrix} a \\ a^2 \end{bmatrix}$ and c is any real number, then

$$c \odot \mathbf{u} = \begin{bmatrix} ca \\ ca^2 \end{bmatrix} \neq \begin{bmatrix} ca \\ (ca)^2 \end{bmatrix}$$

and so $c \odot \mathbf{u}$ is not in the proposed subspace.

(c) Are the integers a real vector space (with ordinary addition and scalar multiplication)? Justify your answer.

The integers are not a real vector space. They satisfy all necessary conditions except for closure under scalar multiplication. For example, if we let $c = \pi$, then $c \odot \mathbf{u}$ is not an integer for any integer $\mathbf{u}$.