Linear Algebra I **Test 1** **October 25, 2004**

1. Use the Gram-Schmidt process on $\mathbf{R}^3$ to orthonormalize the basis $\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\-1\\0\end{bmatrix}, \begin{bmatrix}3\\-2\\5\end{bmatrix}$.

(Remember to clear fractions as you go and then normalize at the end.)

Call the given vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$. Set $\mathbf{v}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$. Then

$$\mathbf{v}_2 = \mathbf{u}_2 - \frac{(\mathbf{u}_2, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 = \begin{bmatrix}0\\-1\\0\end{bmatrix} - \frac{-1}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}1/3\\-2/3\\1/3\end{bmatrix}.$$

For convenience, we multiply by three to obtain

$$\mathbf{v}_2 = \begin{bmatrix}1\\-2\\1\end{bmatrix}.$$

Now we have

$$\mathbf{v}_3 = \mathbf{u}_3 - \frac{(\mathbf{u}_3, \mathbf{v}_2)}{(\mathbf{v}_2, \mathbf{v}_2)}\mathbf{v}_2 - \frac{(\mathbf{u}_3, \mathbf{v}_1)}{(\mathbf{v}_1, \mathbf{v}_1)}\mathbf{v}_1 = \begin{bmatrix}3\\-2\\5\end{bmatrix} - \frac{12}{6}\begin{bmatrix}1\\-2\\1\end{bmatrix} - \frac{6}{3}\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

Thus an orthonormal basis is

$$\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}, \frac{1}{\sqrt{6}}\begin{bmatrix}1\\-2\\1\end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix}-1\\0\\1\end{bmatrix}.$$

2. Find a basis for $\mathbf{R}^3$ that includes the vectors $\begin{bmatrix}2\\-2\\6\end{bmatrix}$ and $\begin{bmatrix}0\\1\\-3\end{bmatrix}$.

We form the matrix

$$\begin{bmatrix}2 & 0 & 1 & 0 & 0\\-2 & 1 & 0 & 1 & 0\\6 & -3 & 0 & 0 & 1\end{bmatrix}$$

and row reduce to obtain

$$\begin{bmatrix}2 & 0 & 1 & 0 & 0\\0 & 1 & 1 & 1 & 0\\0 & 0 & 0 & 3 & 1\end{bmatrix}.$$

If we divide the first row by 2 and the third by 3, we see that the initial 1s lie in columns 1, 2 and 4. Thus a basis is

$$\begin{bmatrix}2\\-2\\6\end{bmatrix}, \begin{bmatrix}0\\1\\-3\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}.$$

3. (a) Suppose $S = \{t, t-3, t^2+1\}$ and $T = \{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ are ordered bases for P_2, and suppose the transition matrix from T to S is $P_{S\leftarrow T} = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Find the basis T.

The jth column of $P_{S\leftarrow T}$ is the vector $[\mathbf{w}_j]_S$. Thus we have

$$\mathbf{w}_1 = 2\mathbf{v}_1 + 1\mathbf{v}_2 + 0\mathbf{v}_3 = 2t + t - 3 = 3t3,$$
$$\mathbf{w}_2 = 4\mathbf{v}_1 - 1\mathbf{v}_2 + 1\mathbf{v}_3 = 4t - (t-3) + t^2 + 1 = t^2 + 3t + 4,$$
$$\mathbf{w}_3 = 1\mathbf{v}_1 + 0\mathbf{v}_2 + 1\mathbf{v}_3 = t + t^2 + 1 = t^2 + t + 1.$$

(b) Use $P_{S\leftarrow T}$ to find $[\mathbf{v}]_S$, given that $[\mathbf{v}]_T = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$.

This is the definition of what the transition matrix does, namely, $P_{S\leftarrow T}[\mathbf{v}]_T = [\mathbf{v}]_S$. Thus

$$[\mathbf{v}]_S = \begin{bmatrix} 2 & 4 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

4. State the Cauchy-Schwarz inequality for an arbitrary vector space V with inner product $(\ ,\ )$.

$$|(\mathbf{u}, \mathbf{v})| \leq \| \mathbf{u} \| \| \mathbf{v} \|$$

5. Find the Fourier polynomial of degree two for the function $f(t) = \cos^2(t)$ with respect to the orthonormal basis

$$\left\{ \frac{1}{\sqrt{2\pi}},\ \frac{1}{\sqrt{\pi}}\sin(t),\ \frac{1}{\sqrt{\pi}}\cos(t),\ \frac{1}{\sqrt{\pi}}\sin(2t),\ \frac{1}{\sqrt{\pi}}\cos(2t),\ \ldots \right\}$$

using the following facts:

$$\int_{-\pi}^{\pi} \cos^2(t)\,dt = \pi \qquad \int_{-\pi}^{\pi} \cos^2(t)\sin(t)\,dt = 0 \qquad \int_{-\pi}^{\pi} \cos^3(t)\,dt = 0$$
$$\int_{-\pi}^{\pi} \cos^2(t)\sin(2t)\,dt = 0 \qquad \int_{-\pi}^{\pi} \cos^2(t)\cos(2t)\,dt = \pi/2$$

We know that we can write

$$\cos^2(t) \approx a_0\frac{1}{\sqrt{2\pi}} + a_1\frac{1}{\sqrt{\pi}}\sin(t) + b_1\frac{1}{\sqrt{\pi}}\cos(t) + a_2\frac{1}{\sqrt{\pi}}\sin(2t) + b_2\frac{1}{\sqrt{\pi}}\cos(2t)$$

where the coefficients are expressed in terms of the given integrals. Because most of these integrals are zero, we have

$$\cos^2(t) \approx a_0\frac{1}{\sqrt{2\pi}} + b_2\frac{1}{\sqrt{\pi}}\cos(2t).$$

Now

$$a_0 = \int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} \cos^2(t)\, dt = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} \cos^2(t)\, dt = \frac{\pi}{\sqrt{2\pi}}.$$

Also,

$$b_2 = \int_{-\pi}^{\pi} \frac{1}{\sqrt{\pi}} \cos^2(t) \cos(2t)\, dt = \frac{\pi}{2\sqrt{pi}}.$$

Thus we obtain

$$\cos^2(t) \approx \frac{\pi}{\sqrt{2\pi}} \frac{1}{\sqrt{2\pi}} + \frac{\pi}{2\sqrt{\pi}} \frac{1}{\sqrt{\pi}} \cos(2t) = \frac{1}{2} + \frac{1}{2}\cos(2t).$$

Note that this is actually equality, as we have discovered a double angle formula the hard way.

6. Suppose we define a function $L\colon P_4 \to M_{22}$ by saying that

$$L(at^3 + bt^2 + ct + d) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Verify that L is an isomorphism by checking that it preserves the vector space operations (you may assume it is one-to-one and onto).

Pick two vectors in P_4, say $\mathbf{u} = at^3 + bt^2 + ct + d$ and $\mathbf{v} = \alpha t^3 + \beta t^2 + \gamma t + \delta$. We need first to show that $L(\mathbf{u} + \mathbf{v}) = L(\mathbf{u}) + L(\mathbf{v})$. To this end, note that

$$L(\mathbf{u} + \mathbf{v}) = L(at^3 + bt^2 + ct + d + \alpha t^3 + \beta t^2 + \gamma t + \delta) =$$

$$L((a+\alpha)t^3 + (b+\beta)t^2 + (c+\gamma)t + (d+\delta)) = \begin{bmatrix} a+\alpha & b+\beta \\ c+\gamma & d+\delta \end{bmatrix}.$$

On the other hand, we have

$$L(\mathbf{u}) + L(\mathbf{v}) = L(at^3 + bt^2 + ct + d) + L(\alpha t^3 + \beta t^2 + \gamma t + \delta) = \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}.$$

These matrices are equal, so we have verified the first property.

For the second property, we need to show that $L(r\mathbf{u}) = rL(\mathbf{u})$, for any real number r. But we have

$$L(r\mathbf{u}) = L(r(at^3 + bt^2 + ct + d)) = L(rat^3 + rbt^2 + rct + rd) = \begin{bmatrix} ra & rb \\ rc & rd \end{bmatrix},$$

while

$$rL(\mathbf{u}) = rL(at^3 + bt^2 + ct + d) = r \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Again, these matricies are equal, so L is an isormorphism.

7. Suppose V is an inner product space. Show that if $\mathbf{u}$ is orthogonal to each of $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in V, then $\mathbf{u}$ is orthogonal to any vector in the subspace $W = \operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$.

Because $\mathbf{u}$ is orthogonal to each $\mathbf{v}_i$, we know that $(\mathbf{u}, \mathbf{v}_i) = 0$ for each i. Pick a vector $\mathbf{v}$ in W. Then $\mathbf{v} = a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n$. We need to show that $(\mathbf{u}, \mathbf{v}) = 0$. But

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{u}, a_1\mathbf{v}_1 + \cdots + a_n\mathbf{v}_n) = a_1(\mathbf{u}, \mathbf{v}_1) + \cdots + a_n(\mathbf{u}, \mathbf{v}_n) = 0.$$