

1. Suppose $L: V \rightarrow W$ is a linear transformation, where the dimension of V is n and the dimension of W is m .

(a) State the rank-nullity theorem for L .

$$\dim(\ker(L)) + \dim(\text{range}(L)) = n$$

(b) Suppose $n = 3$. If L is one-to-one, what is the dimension of the range of L ?

Because L is one-to-one, the dimension of the kernel of L is zero. The rank-nullity theorem then says that the dimension of the range is $n = 3$.

(c) Suppose $n > m$. Can L be one-to-one? Explain.

L cannot be one-to-one. To see this algebraically, note that the dimension of the range of L must be less than or equal to m (the range of L is that part of W that L hits, so it can't be bigger than all of W). So then we have

$$\dim(\ker(L)) = n - \dim(\text{range}(L)) \geq n - m > 0.$$

Thus the dimension of the kernel is not zero, so L is not one-to-one. More prosaically, the idea is that we start with n dimensions over in V , but then when we apply L we only have m dimensions remaining. Because $n > m$, some dimensions must have died. In other words, L must have nontrivial kernel.

2. Find the determinants of the following matrices:

(a) $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$

$$(1)(2) - (-2)(3) = 8$$

(b) $\begin{bmatrix} 2 & -1 & 2 \\ 3 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

The first and third columns are equal, so the determinant is zero.

(c) $\begin{bmatrix} 1 & 8 & 5 & -3 & 5 \\ 0 & 1 & -2 & -4 & 1 \\ 0 & 0 & 1 & 8 & 7 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

The determinant of an upper triangular matrix is the product of the diagonal elements, which is 1.

3. Use determinants and cofactors to find the inverse of the following matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

The cofactors are

$$\begin{aligned}A_{11} &= -2 & A_{21} &= -(0) & A_{31} &= 0 \\A_{12} &= -(-1) & A_{22} &= -1 & A_{32} &= -(1) \\A_{13} &= -1 & A_{23} &= -(1) & A_{33} &= 1\end{aligned}$$

and the determinant is -2 , so the inverse is

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

4. Verify the rank-nullity theorem for $L: \mathbf{R}^3 \rightarrow \mathbf{R}^2$, where

$$L \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + 3c \\ 2b + c \end{bmatrix},$$

by finding a basis for both the kernel and the range of L .

For $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to be in the kernel of L , we must have

$$L \left(\begin{bmatrix} a \\ b \\ c \end{bmatrix} \right) = \begin{bmatrix} a + 3c \\ 2b + c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

Solving, we find that $c = -2b$ and $a = -3c = 6b$. So the kernel is all vectors of the form

$$\begin{bmatrix} 6b \\ b \\ -2b \end{bmatrix} = b \begin{bmatrix} 6 \\ 1 \\ -2 \end{bmatrix}.$$

This vector therefore spans the kernel, which is thus one dimensional.

The range of L consists of all vectors of the form

$$\begin{bmatrix} a + 3c \\ 2b + c \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 2 \end{bmatrix} + c \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

These three vectors therefore span the range. These three are linearly dependent, however (for example, the last one is clearly a linear combination of the first two). On the other hand, no one of them is a multiple of any other, so any two are linearly independent. Thus the dimension of the range is two.

The domain of L is \mathbf{R}^3 , which is three-dimensional. As $1 + 2 = 3$, we have verified the rank-nullity theorem.

5. Suppose $L: \mathbf{R}^3 \rightarrow \mathbf{R}^3$ is a linear map. Let T be the basis

$$T = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\},$$

and suppose

$$L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad L\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix}.$$

Find the matrix representation for L with respect to T and S . (Hint: Use the fact that L is linear to find the image under L of the T vectors.)

Method 1: Use the hint. To find the matrix representation for L with respect to T and S , we want to apply L to the T vectors and then express the results in terms of the S basis. This last bit is trivial, because S is the standard basis, so all we need to do is apply L to the T basis vectors. We follow the hint and use the fact that L is linear to obtain

$$L\left(\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = L\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right) + L\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}.$$

Similarly we find that

$$L\left(\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 4 \\ 5 \end{bmatrix} \quad L\left(\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

Thus the matrix we seek is

$$\begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 2 \\ 1 & 5 & 4 \end{bmatrix}$$

Method 2: Ignore the hint. This way is computationally just as easy, but hurts the brain a little bit more. The idea is to draw the giant diagram, with the matrix A representing L with respect to S on the bottom, and the matrix B representing L with respect to T and S on the top. From the information given, we have that

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix}.$$

If you look at the diagram, you see that to get B , we multiply A on the right by the transition matrix from T to S and on the left by the transition matrix from S to S . Now the transition matrix from T to S is just

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix},$$

while the transition matrix from S to S is just the identity matrix. Thus we have

$$B = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 1 & 3 \\ 0 & 1 & 4 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 1 & 2 \\ 0 & 4 & 2 \\ 1 & 5 & 4 \end{bmatrix}.$$

6. Suppose $L: P_2 \rightarrow P_2$ is defined by

$$L(p(t)) = tp'(t) + 6 \int_0^1 p(t) dt.$$

(a) Suppose $S = \{t^2, t, 1\}$ is the standard basis for P_2 . Find the matrix representation for L with respect to the basis S .

To find the columns of the matrix we seek, we apply L to each of the S vectors, and then find the coordinates for the results in terms of the S basis. We have that $L(t^2) = 2t^2 + 2$, $L(t) = t + 3$, and $L(1) = 6$, the S -coordinates for which are

$$\begin{bmatrix} 2 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 6 \end{bmatrix},$$

so the matrix we seek is

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 6 \end{bmatrix}.$$

(b) Use transition matrices and your answer from part (a) to find the matrix representation for L with respect to the basis

$$T = \{t^2 + 1, t + 1, t - 1\}.$$

(Hint: Your work in problem #3 will be helpful here.)

If we let A be the matrix found in part (a), then the matrix we seek is $B = (P_{S \leftarrow T})^{-1}AP_{S \leftarrow T}$. The columns of $P_{S \leftarrow T}$ are the S -coordinates of the T basis vectors. Thus

$$P_{S \leftarrow T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix}.$$

The inverse of this matrix was found in problem #3. So we find that

$$B = \frac{1}{2} \begin{bmatrix} 2 & 0 & 0 \\ -1 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 6 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 3 & 5 & -1 \\ -3 & -4 & 2 \end{bmatrix}.$$