Calculus III   Test 1 Solutions   September 15, 2005

1. Sketch the curve traced out by the parameter in the following parametric equations:

$$x = \sin t, \qquad y = 1 - \cos^2 t.$$

Be sure to describe the way in which the curve is traced out as the parameter increases. (You may find it useful to eliminate the parameter to find a Cartesian equation describing the curve.)

Because $1 - \cos^2 t = \sin^2 t$, the path traced out lies along the parabola $y = x^2$. Because $x = \sin t$ is always between -1 and 1, we only ever trace out that portion of the parabola between $x = -1$ and $x = 1$. As t increases, this portion is traced out repeatedly back and forth, with $t = -\pi/2$ corresponding to the far left endpoint, $t = 0$ the origin, and $t = \pi/2$ the far right endpoint.

2. Determine whether the sequence converges or diverges, being sure to justify your claims. If it converges, find the limit.

(a) $a_n = \dfrac{\ln n}{n}$

Consider the function $f(x) = \dfrac{\ln x}{x}$. Then

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

It follows from the connect-the-dots theorem that the sequence also converges to zero.

(b) $a_n = \dfrac{\cos^2(n)}{2^n}$

Note that

$$0 \le a_n \le \frac{1}{2^n},$$

but

$$\lim_{n\to\infty} \frac{1}{2^n} = 0,$$

so the sequence converges to zero, by the squeeze theorem.

3. (a) Sketch the region described by the following polar inequalities:

$$-1 \le r \le 1, \qquad \pi/4 \le \theta \le 3\pi/4.$$

This is the region inside the unit circle between the angles of $\pi/4$ and $3\pi/4$ on top, and between the angles of $5\pi/4$ and $7\pi/4$ on bottom.

(b) Find Cartesian coordinates for the point with polar coordinates $(2\sqrt{2}, 3\pi/4)$.

We use the fact that $x = r\cos\theta$ and $y = r\sin\theta$ to obtain

$$x = (2\sqrt{2})\Big(-\frac{\sqrt{2}}{2}\Big) = -2$$
$$y = (2\sqrt{2})\Big(\frac{\sqrt{2}}{2}\Big) = 2,$$

so the point is

$$(x, y) = (-2, 2).$$

(c) Find two more sets of polar coordinates describing the point from part (b), at least one of which has $r < 0$.

We may add 2π to the angle without changing anything, or we may add π to the angle and change the sign of r. Thus we have, for example,

$$(2\sqrt{2}, 11\pi/4), \qquad (-2\sqrt{2}, 7\pi/4).$$

4. Consider the following parametrized curve:

$$x = t^2 \qquad y = t^3 - 3t.$$

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

Note that $\dfrac{dy}{dt} = 3t^2 - 3$ and $\dfrac{dx}{dt} = 2t$, so that

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}.$$

For the second derivative, note that

$$\frac{d}{dt}\left(\frac{dy}{dx}\right) = \frac{(6t)(2t) - 2(3t^2 - 3)}{(2t)^2},$$

so that

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{12t^2 - 6t^2 + 6}{(2t)^3} = \frac{3t^2 + 3}{4t^3}.$$

(b) Find all values of t at which the tangent to the curve is horizontal and all values of t at which the tangent is vertical.

First note that $\dfrac{dy}{dt} = 3t^2 - 3$ is zero when $t = \pm 1$, while $\dfrac{dx}{dt} = 2t$ is zero when $t = 0$. As there is no overlap in these t values, the path is horizontal when $t = \pm 1$ and vertical when $t = 0$.

(c) Set up, but do not evaluate, an integral equal to the length of the piece of the curve joining $(0, 0)$ to $(4, 2)$.

Note that $(x, y) = (0, 0)$ implies that $t = 0$, while $(x, y) = (4, 2)$ implies that $t = 2$. Thus we have

$$L = \int_0^2 \sqrt{\left(\frac{dy}{dt}\right)^2 + \left(\frac{dx}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(3t^2 - 3)^2 + (2t)^2}\, dt$$

5. Find the area inside the leminiscate $r^2 = 8 \cos 2\theta$ and outside the circle $r = 2$.

To find the limits of integration, set the functions equal to one another and solve for θ. Thus we have

$$\sqrt{8 \cos 2\theta} = 2 \quad \Rightarrow \quad 8 \cos 2\theta = 4 \quad \Rightarrow \quad \cos 2\theta = \frac{1}{2} \quad \Rightarrow \quad 2\theta = \pi/3 \quad \Rightarrow \quad \theta = \pi/6.$$

Of course there are other values of θ solving the original equation, but they can all be determined using the symmetry of the picture. In particular, one way of computing the area is as follows:

$$A = 4\left[\frac{1}{2}\int_0^{\pi/6} 8 \cos 2\theta\, d\theta - \frac{1}{2}\int_0^{\pi/6} 4\, d\theta\right] = \left[16\left(\frac{\sin 2\theta}{2}\right) - 8\theta\right]_0^{\pi/6} = 4\sqrt{3} - \frac{4\pi}{3}.$$

6. Determine whether the series converges or diverges, being sure to justify your claims. If possible, find the sum.

(a) $\frac{1}{3} + \frac{\pi}{9} + \frac{\pi^2}{27} + \frac{\pi^3}{81} + \cdots$

Note that

$$\frac{\pi/9}{1/3} = \frac{\pi^2/27}{\pi/9} = \frac{\pi^3/81}{\pi^2/27} = \frac{\pi}{3},$$

so that this series is geometric with ratio $r = \frac{\pi}{3}$. Because $\frac{\pi}{3} > 1$, this series diverges.

(b) $\ln(1) + \ln(1/2) + \ln(1/3) + \ln(1/4) + \cdots$

Note that

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \ln\left(\frac{1}{n}\right) = \ln\left(\lim_{n\to\infty} \frac{1}{n}\right) \to -\infty,$$

so the series diverges by the divergence test.

(c) $5 - \frac{10}{3} + \frac{20}{9} - \frac{40}{27} + \cdots$

Note that

$$\frac{-10/3}{5} = \frac{20/9}{-10/3} = \frac{-40/27}{20/9} = \frac{2}{3},$$

so the series is geometric with $r = -\frac{2}{3}$. Because $|r| < 1$, the series converges to

$$\frac{5}{1 - \frac{-2}{3}} = 3.$$