

Calculus III Test 2 Solutions October 25, 2005

1. (a) Use known series to find a Taylor series (centered at $x = 0$) for the following indefinite integral. Either write the series in summation notation or find at least the first five non-zero terms of the series. Also state the radius of convergence.

$$\int \frac{x^2}{1+2x} dx$$

Note that

$$\begin{aligned} \frac{x^2}{1+2x} &= x^2 \frac{1}{1-(-2x)} = x^2(1 + (-2x) + (-2x)^2 + (-2x)^3 + (-2x)^4 + \dots) \\ &= x^2 - 2x^3 + 2^2x^4 - 2^3x^5 + 2^4x^6 - \dots = \sum_{n=0}^{\infty} (-1)^n (2^n) x^{n+2}. \end{aligned}$$

Thus by integrating we obtain

$$\int \frac{x^2}{1+2x} dx = C + \frac{x^3}{3} - \frac{2x^4}{4} + \frac{2^2x^5}{5} - \frac{2^3x^6}{6} + \frac{2^4x^7}{7} - \dots = C + \sum_{n=0}^{\infty} (-1)^n \frac{2^n x^{n+3}}{n+3}.$$

This series converges as long as $|-2x| < 1$, or $|x| < \frac{1}{2}$, thus the radius of convergence is $1/2$.

(b) Use Taylor series to verify that the derivative of $\sin x$ is $\cos x$.

$$\frac{d}{dx} \sin x = \frac{d}{dx} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = \cos x,$$

or

$$\frac{d}{dx} \sin x = \frac{d}{dx} \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \dots \right) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \dots = \cos x.$$

2. (a) Write the formula for the Taylor series for $f(x)$ centered at $x = a$.

$$T(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

(b) Find the degree two Taylor approximation for \sqrt{x} centered at $x = 9$, and use this approximation to estimate $\sqrt{10}$.

The degree two Taylor polynomial is the beginning of the Taylor series for \sqrt{x} at $x = 9$, up to the $(x-a)^2$ term. Note that $f(9) = 3$, $f'(9) = 1/6$, and $f''(9) = -1/108$. Thus we have

$$T_2(x) = 3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2.$$

Plugging in $x = 10$ to this, we obtain the approximation

$$\sqrt{10} \approx 3 + \frac{1}{6} - \frac{1}{216} = \frac{683}{216} \approx 3.162.$$

(c) Use the Taylor series for $f(x) = \cos(x^3)$ (centered at zero) to find $f^{(12)}(0)$.

Substituting x^3 into the series for cosine, we have

$$\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} = 1 - \frac{x^6}{2} + \frac{x^{12}}{4!} - \dots$$

Thus the coefficient of x^{12} is $\frac{1}{4!}$. On the other hand, Taylor's formula says that this is equal to $f^{(12)}(0)/(12!)$. Thus we have

$$\frac{f^{(12)}(0)}{12!} = \frac{1}{4!} \quad \Rightarrow \quad f^{(12)}(0) = \frac{12!}{4!} = 12 \cdot 11 \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 = 19\,958\,400.$$

3. Find the interval of convergence for the following series:

$$\sum_{n=0}^{\infty} \frac{(x-3)^n}{2^n \sqrt{n+3}} = \frac{1}{\sqrt{3}} + \frac{(x-3)}{2\sqrt{4}} + \frac{(x-3)^2}{4\sqrt{5}} + \frac{(x-3)^3}{8\sqrt{6}} + \frac{(x-3)^4}{16\sqrt{7}} + \dots$$

We use the ratio test:

$$\lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{(x-3)^{n+1} 2^n \sqrt{n+3}}{(x-3)^n 2^{n+1} \sqrt{n+4}} \right| = \left| \frac{x-3}{2} \right| \lim_{n \rightarrow \infty} \left| \sqrt{\frac{n+3}{n+4}} \right| = \left| \frac{x-3}{2} \right|.$$

The ratio test says that as long as this limit is less than one, the series will converge. So we have

$$\left| \frac{x-3}{2} \right| < 1 \quad \Rightarrow \quad -1 < \frac{x-3}{2} < 1 \quad \Rightarrow \quad -2 < x-3 < 2 \quad \Rightarrow \quad 1 < x < 5.$$

So the series converges for $1 < x < 5$ and diverges for $x < 1$ and $x > 5$. We still need to check the endpoints.

Let $x = 1$. Then the series becomes

$$\sum_{n=0}^{\infty} \frac{(1-3)^n}{2^n \sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^n}{2^n \sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n+3}}.$$

This series converges by the alternating series test (note that as $n \rightarrow \infty$, the denominators increase, so the fractions decrease, and the limit is zero).

Now let $x = 5$. Then the series becomes

$$\sum_{n=0}^{\infty} \frac{2^n}{2^n \sqrt{n+3}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n+3}}.$$

We expect this series to diverge because it is essentially the same as the divergent p -series $\sum \frac{1}{\sqrt{n}}$. Because the necessary inequality goes the wrong way, we need to use the limit comparison test. Thus we have

$$\lim_{n \rightarrow \infty} \frac{\sqrt{n}}{\sqrt{n+3}} = 1.$$

Because the limit is neither zero nor infinite, the two series converge or diverge together. So because the p -series diverges, our series diverges.

The interval of convergence is therefore

$$1 \leq x < 5.$$

4. Determine whether each of the following series converges or diverges.

$$(a) \sum_{n=1}^{\infty} \frac{\cos^2 n}{n\sqrt{n}} = \cos^2(1) + \frac{\cos^2(2)}{2\sqrt{2}} + \frac{\cos^2(3)}{3\sqrt{3}} + \frac{\cos^2(4)}{4\sqrt{4}} + \frac{\cos^2(5)}{5\sqrt{5}} + \dots$$

We use the comparison test, noting that

$$0 < \frac{\cos^2 n}{n\sqrt{n}} < \frac{1}{n\sqrt{n}} = \frac{1}{n^{3/2}}.$$

Because our series is positive and smaller than the convergent p -series $\sum \frac{1}{n^{3/2}}$, our series converges.

$$(b) \sum_{n=1}^{\infty} \left(\frac{3n}{1+8n}\right)^n = \frac{3}{9} + \left(\frac{6}{17}\right)^2 + \left(\frac{9}{25}\right)^3 + \left(\frac{12}{33}\right)^4 + \left(\frac{15}{41}\right)^5 + \dots$$

We use the root test, and note that

$$\lim_{n \rightarrow \infty} \sqrt[n]{|a_n|} = \lim_{n \rightarrow \infty} \frac{3n}{1+8n} = \frac{3}{8} < 1,$$

so that the series converges.

5. Find the sum of the following series:

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = 1 + \frac{3}{5} + \frac{3^2}{5^2 2!} + \frac{3^3}{5^3 3!} + \frac{3^4}{5^4 4!} + \dots$$

Note that

$$\sum_{n=0}^{\infty} \frac{3^n}{5^n n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}.$$