Calculus III Test 3 Solutions November 22, 2005

1. Sketch the traces of the surface described by the equation below. Each trace graph should have at least three different curves on it.

$$x^2 + y - z^2 = 0$$

The x-traces are parabolas centered on the y-axis, opening up, and shifting down as x increases in absolute value. The y-traces are hyperbolas. The z-traces are parabolas centered on the y-axis, opening down, and shifting up as z increases in absolute value.

2. Matching:

$x^2 + 4y^2 + 9z^2 = 1$ is G, an ellipsoid

$y^2 = x^2 + z^2 - 1$ is L, a hyperboloid of one sheet

$y = 2x^2 + z^2$ is J, an elliptic paraboloid

$x^2 + 2z^2 = 1$ is A, a cylinder centered on the y-axis

$y^2 = x^2 + z^2 + 1$ is H, a hyperboloid of two sheets

$y^2 = x^2 + 2z^2$ is M, a double cone

$y = x^2 - z^2$ is I, a hyperbolic paraboloid (saddle)

$r = 2$ is C, a cylinder centered on the z-axis

$\rho = 2$ is O, a sphere

$\theta = \pi/3$ is B, a half-plane

$\phi = \pi/3$ F, a single cone

3. Consider the vectors $\mathbf{a} = \langle 3, 1, 0\rangle$ and $\mathbf{b} = \langle -2, 1, 2\rangle$. Find the following (angles may be left in terms of inverse trig functions if necessary):

(a) $\mathbf{a} \cdot \mathbf{b} = -5$

(b) $\mathbf{a} \times \mathbf{b} = \langle 2, -6, 5\rangle$

(c) the (cosine of the) angle between $\mathbf{a}$ and $\mathbf{b}$ is $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \dfrac{-5}{3\sqrt{10}}$

(d) the projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\dfrac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = -\frac{1}{2}\langle 3, 1, 0\rangle$

(e) the area of the triangle with $\mathbf{a}$ and $\mathbf{b}$ as two of its sides is $\frac{1}{2}|\mathbf{a} \times \mathbf{b}| = \frac{1}{2}\sqrt{65}$

(f) an equation for the plane through $(1, 2, 3)$ and parallel to both $\mathbf{a}$ and $\mathbf{b}$: the normal is $\mathbf{a} \times \mathbf{b} = \langle 2, -6, 5\rangle$, so the plane is given by $2(x - 1) - 6(y - 2) + 5(z - 3) = 0$, or $2x - 6y + 5z = 5$.

(g) the (cosine of the) angle between the plane through $(1, 2, 3)$ with normal vector $\mathbf{a}$ and the plane through $(1, 2, 3)$ with normal vector $\mathbf{b}$: the angle between the planes is the same as that between their normals, which was found in part (c) to have cosine equal to $\dfrac{-5}{3\sqrt{10}}$.

(h) parametric equations for the line of intersection between the two planes in part (g): the direction vector is contained in both planes, and so perpendicular to both normals, and so parallel to their cross product $\langle 2, -6, 5\rangle$. We are told that $(1, 2, 3)$ is contained in both planes, so we use that as our point. Thus we have $\mathbf{r}(t) = \langle 1 + 2t, 2 - 6t, 3 + 5t\rangle$.

(i) the volume of the parallelepiped three of whose sides are $\mathbf{a}$, $\mathbf{b}$, and $\langle 1, 2, 3\rangle$ is $|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = |(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 5$

4. Let $\mathbf{a}$ and $\mathbf{b}$ be the same vectors as above.

(a) Suppose $\mathbf{b}$ represents a force vector (in Newtons) acting on an object at the point $(1, 2, 3)$, moving it in the direction of $\mathbf{a}$ a distance of 5 meters. Find the work done by this force.

We use the formula $W = \mathbf{F} \cdot \mathbf{D}$. Applying this directly, we have that $\mathbf{F} = \langle -2, 1, 2\rangle$, and $\mathbf{D} = \frac{5}{\sqrt{10}}\langle 3, 1, 0\rangle$ (we divide $\mathbf{a}$ by its length to make it unit length, then multiply by five to make it length five). Thus $W = -\dfrac{25}{\sqrt{10}}$. Another way to approach the problem is to use the formula $\mathbf{F} \cdot \mathbf{D} = |\mathbf{F}||\mathbf{D}| \cos\theta$, noting that $|\mathbf{F}| = 3$, $|\mathbf{D}| = 5$, and using the cosine that was found in part (c) above.

(b) Suppose there is a wrench fixed on a bolt at the origin whose handle points in the direction of the positive y axis. (The bolt is aligned so that the wrench will turn around in the xy-plane.) Suppose a force of 50 Newtons is applied at some point along this wrench at an angle of $2\pi/3$, measured counterclockwise in the xy-plane from the positive x-axis in the usual way. How far along the wrench from the bolt should this force be applied in order to supply 100 Joules of torque to the bolt?

We use the formula $|\tau| = |\mathbf{r} \times \mathbf{F}| = |\mathbf{r}||\mathbf{F}| \sin\theta$. We want to have $|\tau| = 100$ and we are given that $|\mathbf{F}| = 50$. Note that the angle between $\mathbf{r}$ and $\mathbf{F}$ is $\pi/6$, which has sine equal to $\frac{1}{2}$. It follows that $|\mathbf{r}| = 4$.