

Answers to homework 20 problems

17.8 #2: We use Stokes' theorem to calculate

$$\iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \int_C \mathbf{F} \cdot d\mathbf{r},$$

where  $C$  is the boundary of  $S$ , endowed with the appropriate orientation. Note that  $C$  is a circle of radius 2 lying in the plane  $z = 5$ , and we need the parametrization to run counterclockwise (viewed from above). We thus have the parametrization  $\mathbf{r}(t) = \langle 2 \cos t, 2 \sin t, 5 \rangle$ . Now we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt = \int_0^{2\pi} \langle 10 \sin t, 10 \cos t, 4 \cos t \sin t \rangle \cdot \langle -2 \sin t, 2 \cos t, 0 \rangle dt \\ &= \int_0^{2\pi} (-20 \sin^2 t + 20 \cos^2 t) dt = 20 \int_0^{2\pi} \cos(2t) dt = 0, \end{aligned}$$

where the second to last equality is a double angle formula.

17.8 #9: Here we use Stokes' theorem in the other direction, replacing a line integral with a surface integral. Thus we need a surface  $S$  whose boundary is the given curve  $C$ . The simplest choice is the disk  $x^2 + y^2 \leq 16$ ,  $z = 5$ . Since  $C$  is oriented counterclockwise (from above), we orient  $S$  upward. Because  $S$  is part of a plane, we use parametrization  $\mathbf{r} = \langle u, v, 5 \rangle$ , over the disk  $D$  of radius 4 in the  $uv$ -plane, and the upward pointing normal is simply  $\langle 0, 0, 1 \rangle$ . Thus we have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \operatorname{curl} \mathbf{F}(\mathbf{r}(u, v)) \cdot \langle 0, 0, 1 \rangle dA \\ &= \iint_D \langle ue^{uv} - 2u, -ve^{uv} + v, 5 \rangle \cdot \langle 0, 0, 1 \rangle dA = 5 \iint_D dA = 5(\operatorname{area}(D)) = 80\pi. \end{aligned}$$

17.8 #12a: Note that  $S$  is the part of the surface  $z = x^2 - y^2$  lying over the unit disk. Thus we use the parametrization  $\mathbf{r}(u, v) = \langle u, v, u^2 - v^2 \rangle$  with domain  $D$  the unit disk  $u^2 + v^2 \leq 1$  in the  $uv$ -plane. This has downward pointing normal vector  $-(\mathbf{r}_u \times \mathbf{r}_v) = \langle 2u, -2v, -1 \rangle$ . The curl of  $\mathbf{F}$  is  $\langle x, -y, 0 \rangle$ . We thus have

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} = \iint_D \langle u, -v, 0 \rangle \cdot \langle 2u, -2v, -1 \rangle dA \\ &= \iint_D 2u^2 + 2v^2 dA = 2 \int_0^{2\pi} \int_0^1 r^2 r dr d\theta = \pi. \end{aligned}$$

17.8 #14: The boundary curve has three parts:

$$C_1 : \mathbf{r}(t) = \langle 1 - t, 2t, 0 \rangle \quad C_2 : \mathbf{r}(t) = \langle 0, 2 - 2t, 2t \rangle \quad C_3 : \mathbf{r}(t) = \langle t, 0, 2 - 2t \rangle.$$

Doing all three of these at once, we obtain

$$\begin{aligned} \int_C \mathbf{F} \cdot d\mathbf{r} &= \\ &= \int_0^1 \langle 1 - t, 2t, 0 \rangle \cdot \langle -1, 2, 0 \rangle dt + \int_0^1 \langle 0, 2 - 2t, 0 \rangle \cdot \langle 0, -2, 2 \rangle dt + \int_0^1 \langle t, 0, 0 \rangle \cdot \langle 1, 0, -2 \rangle dt \end{aligned}$$

$$= \int_0^1 10t - 5 dt = 0.$$

To use Stokes' theorem, note that the curl of  $\mathbf{F}$  is  $\langle xz, -yz, 0 \rangle$ , and the plane may be parametrized as  $\mathbf{r}(u, v) = \langle u, v, 2 - 2u - v \rangle$ , over the triangle  $D$  in the  $uv$ -plane, described as  $0 \leq v \leq 2 - 2u$ ,  $0 \leq u \leq 1$ . This surface has normal vector  $\langle 2, 1, 1 \rangle$ . Thus we have

$$\begin{aligned} \iint_S \operatorname{curl} \mathbf{F} \cdot d\mathbf{S} &= \iint_D \langle 2u - 2u^2 - uv, -2v + 2uv + v^2, 0 \rangle \cdot \langle 2, 1, 1 \rangle dA \\ &= \int_0^1 \int_0^{2-2u} (4u - 4u^2 - 2v + v^2) dv du = 0. \end{aligned}$$