Answers to homework 21 problems

17.9 #4: The divergence $\mathbf{F}$ is $3x+1$, so we have

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \int_0^{2\pi}\int_0^2\int_0^{4-r^2}(3r\cos\theta+1)r\,dz\,dr\,d\theta = 8\pi.$$

To compute the surface integral across the boundary, note that we have two surfaces, part of a paraboloid and part of the xy-plane. On the paraboloid, we use the parametrization $\mathbf{r}(u,v) = \langle u, v, 4-u^2-v^2\rangle$, with domain D the disc $u^2+v^2 \leq 4$ of radius 2 in the uv-plane. The outward pointing normal vector is $\mathbf{r}_u \times \mathbf{r}_v = \langle 2u, 2v, 1\rangle$. So we have

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_D \langle u^2, uv, 4-u^2-v^2\rangle \cdot \langle 2u, 2v, 1\rangle\,dA =$$

$$\iint_D 2u(u^2+v^2)+4-u^2-v^2\,du\,dv = \int_0^{2\pi}\int_0^2 2r^3\cos\theta + 4 - r^2\,du\,dv = 8\pi.$$

On the xy-plane, we use parametrization $\mathbf{r}(u,v) = \langle u, v, 0\rangle$ with domain D the disc $u^2+v^2 \leq 4$. The outward pointing normal vector is $\langle 0,0,-1\rangle$. We therefore have

$$\iint_{S_2} \mathbf{F}\cdot d\mathbf{S} = \iint_D \langle u^2, uv, 0\rangle \cdot \langle 0,0,-1\rangle\,dA = \iint_D 0\,dA = 0.$$

The total surface integral is therefore, as expected, equal to

$$8\pi + 0 = 8\pi.$$

17.9 #10: Note that $\operatorname{div}\mathbf{F} = 0$, so

$$\iint_S \mathbf{F}\cdot d\mathbf{S} = \iint_E \operatorname{div}\mathbf{F}\,dV = 0.$$

17.9 #20: Let S be the part of the paraboloid $x^2+y^2+z=2$ lying above the plane $z=1$. Now we create a closed surface by setting S_1 to be the disk $x^2+y^2 \leq 1$ in the plane $z=1$. Then by the divergence theorem, we have

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iint_S \mathbf{F}\cdot d\mathbf{S} + \iint_{S_1} \mathbf{F}\cdot d\mathbf{S}.$$

Because the divergence of $\mathbf{F}$ is 1, we have that

$$\iiint_E \operatorname{div}\mathbf{F}\,dV = \iiint_E dV = \int_0^1\int_0^{2\pi}\int_1^{1-r^2} r\,dz\,dr\,d\theta = \frac{\pi}{2}.$$

the downward pointing normal for S_1 (using the standard parametrization as part of a plane) is $\langle 0,0,-1\rangle$, whose dot product with $\mathbf{F}$ is equal to -1. Thus over the disk S_1, we have

$$\iint_{S_1} \mathbf{F}\cdot d\mathbf{S} = \iint_{S_1} -1\,dS = -\text{area}(S_1) = -\pi.$$

It follows that the flux across S is

$$\frac{\pi}{2} - (-\pi) = \frac{3\pi}{2}.$$

17.9 #22: Because the unit sphere has outward pointing normal vector $\langle x, y, z\rangle$, we have that

$$2x + 2y + z^2 = \langle 2, 2, z\rangle \cdot \langle x, y, z\rangle,$$

so that

$$\iint_S (2x + 2y + z^2)\, dS = \iint_S \langle 2, 2, z\rangle \cdot \langle x, y, z\rangle\, dS = \iint_S \mathbf{F} \cdot d\mathbf{S},$$

where $\mathbf{F} = \langle 2, 2, z\rangle$. We can therefore apply the divergence theorem, noting that the divergence of $\mathbf{F}$ is 1, to obtain

$$\iint_S (2x + 2y + z^2)\, dS = \iiint_E \operatorname{div} \mathbf{F}\, dV = \iiint_E dV = \operatorname{vol}(E) = \frac{4\pi}{3}.$$