

INTRODUCTION

If D is an object, and f is a function defined at all points of D , then the integral of f over D does the following (with a few simplifications, which one can justify) :

- Chop D into small pieces, each of which has size ΔD
- For each piece D_i of D , pick a point p_i ; then $f(p_i)$ is some number
- Add up the products $f(p_i)\Delta D$ for each i
- Take the limit as $\Delta D \rightarrow 0$, obtaining $\int_D f dD$

In this sense, all integrals do the same thing. When we say below that an integral is “adding up numbers along/across/over D ,” we mean to refer to this process as described above. The differences lie only in the notation, and in the most convenient way of actually calculating such a thing.

Notational conventions:

- We write as many \int symbols as there are dimensions of D
- When D is one-dimensional (a curve C), we use Δs to denote a small piece of C ($\int_C f ds$); when C happens to be a piece of x -axis, we simply use Δx and omit the subscript on the integral ($\int f dx$)
- When D is two-dimensional (a surface S), we use ΔS to denote a small piece of S ($\iint_S f dS$); when S happens to be a piece of the xy -plane, we simply use ΔA ($\iint_D f dA$)
- When D is three-dimensional, it will (for us) automatically be a piece of xyz -space, and we use ΔV to denote a small piece of D ($\iiint_D f dV$)

INTEGRALS OVER CURVES C

An integral $\int_C f ds$ over a curve C simply adds up numbers along the curve, where the numbers are given by the function f . To calculate such an integral, we need to have a parametrization $\mathbf{r}(t)$ ($a \leq t \leq b$) of the curve C . We then compute using the following

$$\int_C f ds = \int_a^b f(\mathbf{r}(t)) |\mathbf{r}'(t)| dt.$$

Special cases:

- When C is a piece of the x -axis, say from a to b , we use x as the parameter, and the expression above becomes

$$\int_C f ds = \int_a^b f(x) dx.$$

One often takes the function f to represent height (measured via the y -axis), so that this integral gives (signed) area under the graph $y = f(x)$.

- When the function f is identically 1, this integral simply adds up all the Δs bits, and hence gives the *arclength* of C . Thus we have

$$\text{Length}(C) = \int_C ds = \int_a^b |\mathbf{r}'(t)| dt.$$

- When C is a curve in the plane, note that we have

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

One often takes the function f to represent height (measured via the z -axis), so that this integral gives (signed) area of a curtain hanging from the curve $f(C)$ down to the xy -plane.

- When C is a curve in space, we have

$$|\mathbf{r}'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}.$$

- In the presence of a vector field \mathbf{F} in the plane or space, we often consider the function f defined as the tangential component of \mathbf{F} along C ; i.e., $f = \mathbf{F} \cdot \mathbf{T}$, where T is the *unit* tangent vector $\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$ to the curve C , so that the integral of this function along C becomes

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \left(\mathbf{F}(\mathbf{r}(t)) \cdot \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} \right) |\mathbf{r}'(t)| dt = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) dt,$$

where the left-most expression is just shorthand notation for the case where this particular function is the one being integrated. This is such a useful thing to do that we simply call it “integrating the vector field along the curve.” It is important to remember that we’re still just integrating a function over a curve. It’s just that when we say that we are “integrating a vector field,” we have a specific function in mind, namely, the one giving the tangential component of the vector field along the curve. When you write out this integral in terms of the components $\mathbf{F} = P\mathbf{i} + Q\mathbf{j} + R\mathbf{k}$ of \mathbf{F} and those of $\mathbf{r}(t) = \langle x(t), y(t), z(t) \rangle$, you obtain the expression (in two dimensions, eliminate all terms with z or R)

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_C \left(P \frac{dx}{dt} + Q \frac{dy}{dt} + R \frac{dz}{dt} \right) dt = \int_C P dx + Q dy + R dz,$$

where the right-most expression is just shorthand notation for the integral in the middle.

- When the vector field \mathbf{F} is in the plane, we can also consider the function f defined as the normal component of \mathbf{F} along C ; i.e., $f = \mathbf{F} \cdot \mathbf{n}$, where n is the *unit* normal vector to the curve C , so that the integral of this function along C becomes

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_a^b (\mathbf{F} \cdot \mathbf{n}) |\mathbf{r}'(t)| dt.$$

This is often called the “flux” of the vector field across the curve. When you write out this integral in terms of the components of \mathbf{F} and of \mathbf{r} , you obtain the expression

$$\int_C \mathbf{F} \cdot \mathbf{n} ds = \int_C \left(P \frac{dy}{dt} - Q \frac{dx}{dt} \right) dt = \int_C P dy - Q dx,$$

where, again, the right-most expression is just shorthand notation for the integral in the middle. (This doesn’t make sense for curves in space, because there are lots of normal vectors to a curve in space; see the next section for 3D flux.)