Calculus IV Test 1 Solutions September 15, 2005

1. Suppose $z = g(u, v, w)$, where $u = u(x, y)$, $v = v(x, y)$, and $w = w(x, y)$. Draw a dependency (tree) diagram and write out the chain rule for $\dfrac{\partial z}{\partial y}$.

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial z}{\partial w}\frac{\partial w}{\partial y}$$

2. The following is a contour map for the function $f(x, y)$.

(a) Estimate the value of f at the point $(-2, 2)$.

$$f(-2, 2) \approx 65$$

(b) Based on the diagram, is $f_x(-2, 2)$ positive or negative? What about $f_y(-2, 2)$? Which is larger (in absolute value)?

$$f_x(-2, 2) > 0, \qquad f_y(-2, 2) < 0, \qquad |f_x(-2, 2)| > |f_y(-2, 2)|$$

(c) Sketch the gradient vectors at the indicated points, keeping in mind their relative lengths.

They should all point roughly toward the origin, each perpendicular to the contour line on which it sits. The closer the next contour line is, the longer the arrow should be.

3. (a) Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{(x-y)^2}{(x+y)^2}$ does not exist.

Along the line $y = 0$ (the x-axis), we have the limit

$$\lim_{x\to 0} \frac{x^2}{x^2} = 1.$$

Along the line $y = x$, we have the limit

$$\lim_{x\to 0} \frac{0}{4x^2} = 0.$$

As these one-variable limits are not equal, the two-variable limit does not exist.

(b) Use polar coordinates to find the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2}$.

Using the substitutions $x = r\cos\theta$ and $y = r\sin\theta$ (so $x^2 + y^2 = r^2$), we obtain

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta + r^3\sin^3\theta}{r^2} = \lim_{r\to 0} r(\cos^3\theta + \sin^3\theta).$$

Using the fact that each of $\cos^3\theta$ and $\sin^3\theta$ is between -1 and 1, we have

$$\lim_{r\to 0} |r(\cos^3\theta + \sin^3\theta)| \le \lim_{r\to 0} |r| = 0.$$

So by the squeeze theorem we have

$$\lim_{(x,y)\to(0,0)} \frac{x^3 + y^3}{x^2 + y^2} = 0.$$

5. (a) Find both first partial derivatives of $f(x, y) = e^x \sin(xy)$ at the point $(0, 2)$.

$$f_x(0,2) = [e^x \sin(xy) + ye^x \cos(xy)]_{(0,2)} = 2$$
$$f_y(0,2) = [xe^x \cos(xy)]_{(0,2)} = 0$$

(b) Estimate $f(0.01, 1.97)$ using a tangent plane, a linearization, or the notion of infinitesimal/approximate change (it's all really the same thing).

We will use the formula for infinitesimal change:

$$dz = \frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy.$$

In this case we have $dx = 0.01$ and $dy = -0.03$. Plugging these, and the partial derivatives above, into the equation we obtain

$$dz = 2(0.01) + 0(-0.03) = 0.02.$$

Thus the value of z increases by approximately 0.02 as we move from $(0, 2)$ to $(0.01, 1.97)$. Because $f(0, 2) = 0$, this means that

$$f(0.01, 1.97) \approx f(0, 2) + dz = 0 + 0.02 = 0.02.$$

6. If x and y are the lengths of two sides of a triangle, and the angle they contain is θ, then the area of the triangle is given by

$$A = \frac{1}{2} xy \sin\theta.$$

At noon the dimensions of the triangle are $x = 20$ in., $y = 24$ in., and $\theta = \pi/6$ rad. At the same time, x is increasing at a rate of 3 in/s, y is decreasing at a rate of 2 in/s, and θ is increasing at a rate of $\dfrac{1}{20}$ rad/s.

(a) Write out the chain rule for $\dfrac{dA}{dt}$.

$$\frac{dA}{dt} = \frac{\partial A}{\partial x}\frac{dx}{dt} + \frac{\partial A}{\partial y}\frac{dy}{dt} + \frac{\partial A}{\partial \theta}\frac{d\theta}{dt}$$

(b) Is the area increasing or decreasing? At what rate? Explain.

We are told that

$$\frac{dx}{dt} = 3, \qquad \frac{dy}{dt} = -2 \qquad \frac{d\theta}{dt} = \frac{1}{20}.$$

We also have that

$$\frac{\partial A}{\partial x} = \frac{1}{2} y \sin\theta, \qquad \frac{\partial A}{\partial y} = \frac{1}{2} x \sin\theta, \qquad \frac{\partial A}{\partial \theta} = \frac{1}{2} xy \cos\theta.$$

Evaluating each of these at $(x, y, \theta) = (20, 24, \pi/6)$, and plugging in to the chain rule formula from part (a), we obtain

$$\frac{dA}{dt} = (6)(3) + (5)(-2) + (120\sqrt{3})(1/20) = 18 - 10 + 6\sqrt{3} = 8 + 6\sqrt{3}.$$

7. The depth of a lake is described by the function $f(x,y) = 200 + \frac{x^3}{4} - \frac{y^2}{3}$ (where x and y describe your location on the surface of the lake with respect to some suitable coordinate axes). You are in a boat at the point with coordinates $(4,3)$.

(a) How deep is the lake where you are?

$$f(4,3) = 200 + 16 - 3 = 213.$$

(b) If you head in the direction of the origin, is the water getting deeper or shallower as you begin to move? Explain.

We want the directional derivative of f in the direction of the origin. For this, first note that

$$\nabla f(4,3) = \left\langle \frac{3x^2}{4}, -\frac{2y}{3} \right\rangle \Big|_{(4,3)} = \langle 12, -2 \rangle.$$

Next, note that the direction of the origin is parallel to $\langle -4, -3 \rangle$. The length of this vector is $\sqrt{4^2 + 3^2} = 5$, so the appropriate unit vector is

$$\mathbf{u} = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle.$$

It follows that the directional derivative is

$$D_{\mathbf{u}}(4,3) = \langle 12, -2 \rangle \cdot \langle -4/5, -3/5 \rangle = -\frac{48}{5} + \frac{6}{5} = -\frac{42}{5} < 0.$$

Thus the depth is decreasing in that direction, so the lake is getting shallower.

(c) Your boat springs a leak. You don't care about getting to shore so much as getting to water shallow enough in which to stand. In which direction should you go so as to get to shallow water the fastest?

The direction in which the lake gets shallower the fastest is the direction in which the depth decreases the fastest, which is the negative of the gradient direction. Thus the answer is

$$\langle -12, 2 \rangle.$$