Calculus IV Test 2 Solutions October 13, 2005

1. Consider the following integral:

$$\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} (9 - x^2 - y^2)\, dy\, dx.$$

Describe the region whose volume this integral represents (graphically and/or verbally). Then find the volume by converting the integral to polar coordinates and evaluating.

The region lies below the graph of $z = 9 - x^2 - y^2$ and over the top half of the circle of radius 2 (centered at the origin) in the xy-plane. To find the integral, we use polar coordinates, obtaining

$$\int_0^{\pi}\int_0^{2} (9 - r^2) r\, dr\, d\theta = \int_0^{\pi} \left[\frac{9r^2}{2} - \frac{r^4}{4}\right]_0^2 d\theta = \int_0^{\pi} 14\, d\theta = 14\pi.$$

2. A lamina occupies the region D bounded by the parabolas $x = 2y^2$ and $x = 1 + y^2$ and lying above the x-axis. The density of the lamina at the point (x, y) is described by $\rho(x, y) = y + 2$. Find the mass of the lamina.

This is a type 2 region, so it is easier to integrate with respect to x first (imagine sideways rectangles). Thus we have

$$\text{mass} = \int_0^1 \int_{2y^2}^{1+y^2} (y + 2)\, dx\, dy = \int_0^1 \Big[yx + 2x\Big]_{2y^2}^{1+y^2} dx = \int_0^1 (2 + y - 2y^2 - y^3)\, dy = \frac{19}{12}.$$

3. Carefully sketch the region for the integral $\int_0^1 \int_{3y}^{3} e^{x^2}\, dx\, dy$. Then reverse the order of integration and evaluate the integral.

The region is a right triangle with hypotenuse going from the origin to the point $(3, 1)$, and with right angle at the point $(3, 0)$. Thus we have that the given integral equals

$$\int_0^3 \int_0^{x/3} e^{x^2}\, dy\, dx = \int_0^3 \Big[ye^{x^2}\Big]_0^{x/3} dx = \frac{1}{3}\int_0^3 xe^{x^2}\, dx = \frac{1}{3}\int_0^9 \frac{1}{2}e^u\, du = \frac{1}{6}(e^9 - 1).$$

4. The region E is bounded by the surface $z = 1 - x^2$, the plane $y = 1 - x$, and the three coordinate planes. Express the integral $\iiint_E f(x, y, z)\, dV$ in three different ways, using $dV = dz\, dx\, dy$, $dV = dy\, dx\, dz$, and $dV = dy\, dz\, dx$.

$$\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x, y, z)\, dz\, dx\, dy = \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x, y, z)\, dy\, dx\, dz$$

$$= \int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z)\, dy\, dz\, dx.$$

5. A solid E lies within the cylinder $x^2 + y^2 = 1$, below the plane $z = 4$, and above the paraboloid $z = 1 - x^2 - y^2$. Evaluate the integral

$$\iiint_E (x^2 + y^2)^{3/2}\, dV.$$

Using cylindrical coordinates, we have that the given integral equals

$$\int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 (r^2)^{3/2}\, r\, dz\, dr\, d\theta = \int_0^{2\pi} \int_0^1 \int_{1-r^2}^4 r^4\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \Big[zr^4\Big]_{1-r^2}^4 dr\, d\theta = \int_0^{2\pi} \int_0^1 (3r^4 + r^6)\, dr\, d\theta = \int_0^{2\pi} \left(\frac{3r^5}{5} + \frac{r^7}{7}\Big|_0^1\right) d\theta$$

$$= \int_0^{2\pi} \frac{26}{35}\, d\theta = \frac{52\pi}{35}.$$

6. Evaluate $\iiint_E z\, dV$ where E is the region between the spheres $x^2+y^2+z^2 = 1$ and $x^2+y^2+z^2 = 4$ in the first octant. (Recall that the *first octant* is the region where $x \geq 0$, $y \geq 0$, and $z \geq 0$.)

We use spherical coordinates to obtain

$$\int_0^{\pi/2} \int_0^{\pi/2} \int_1^2 (\rho\cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2} \int_0^{\pi/2} \left[\frac{\rho^4}{4} \sin\phi\cos\phi\right]_{\rho=1}^2 d\phi\, d\theta$$

$$= \frac{15}{4} \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi\cos\phi\, d\phi\, d\theta = \frac{15}{4} \int_0^{\pi/2} \int_0^1 u\, du\, d\theta$$

$$= \frac{15}{8} \int_0^{\pi/2} d\theta = \frac{15\pi}{16}.$$