Calculus IV Test 3 Solutions November 17, 2005

1. I need to hang a curtain along the side of a stage, to separate the visible part of the stage from the backstage area. With appropriate choice of coordinates, the ceiling can be represented by the graph of the equation $f(x, y) = 8e^y$. The curve on the floor along which I need my curtain to pass is given by $\mathbf{r}(t) = \langle t, \ln(\sec t)\rangle$, where $0 \le t \le \frac{\pi}{3}$. Assume all units are meters. How many square meters of cloth do I need to buy to make the curtain?

We need to integrate $\displaystyle\int_C f\, ds$. To do so, note that $f(\mathbf{r}(t)) = 8e^{\ln(\sec t)} = 8\sec t$. Also, we have $|\mathbf{r}'(t)| = |\langle 1, \tan t\rangle| = \sqrt{1+\tan^2 t} = \sec t$. Thus we have

$$\int_C f\, ds = \int_0^{\pi/3} f(\mathbf{r}(t))|\mathbf{r}'(t)|\, dt = \int_0^{\pi/3} 8\sec^2 t\, dt = (\tan t)|_0^{\pi/3} = 8\sqrt{3}.$$

2. (a) State and prove the fundamental theorem for line integrals (you may do it either for two or three dimensions).

The fundamental theorem says that

$$\int_a^b \nabla f(\mathbf{r}(t))\, d\mathbf{r} = f(\mathbf{r}(b)) - f(\mathbf{r}(a)).$$

To prove it (in two dimensions), we let $\mathbf{r}(t) = \langle x(t), y(t)\rangle$. We then have

$$\int_a^b \nabla f(\mathbf{r}(t))\, d\mathbf{r} = \int_a^b \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right\rangle \cdot \left\langle \frac{dx}{dt}, \frac{dy}{dt}\right\rangle dt = \int_a^b \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}\, dt,$$

which, by the chain rule, is equal to

$$\int_a^b \frac{d}{dt} f(\mathbf{r}(t))\, dt = f(\mathbf{r}(b)) - f(\mathbf{r}(a)),$$

where the last equality is the fundamental theorem of calculus.

(b) Let C be the curve $\mathbf{r}(t) = \langle t^4 - t^3, t\sin(\pi t)\rangle$ for $0 \le t \le 1$. Verify that

$$\int_C (2e^y\cos(2x) + 2xy)\, dx + (e^y \sin(2x) + x^2)\, dy = 0,$$

by showing that this is the integral of a conservative vector field along a closed loop. *Do not find a potential function for the vector field.*

To check that $\mathbf{F}$ is conservative, we note that

$$\frac{\partial P}{\partial y} = 2e^y\cos(2x) + 2x = \frac{\partial Q}{\partial x},$$

and also note that $\mathbf{F}$ is defined everywhere in the plane (which is simply connected). Also, because

$$\mathbf{r}(0) = \langle 0, 0\rangle = \mathbf{r}(1),$$

the loop is closed. The result follows.

3. (a) Find a potential function for $\mathbf{F}(x,y) = \langle \cos y, 3y^2 - x\sin y\rangle$.

If $f(x,y)$ is a potential function for $\mathbf{F}$, then $f_x = \cos y$. Thus $f = x\cos y + g(y)$ for some function g. But then $f_y = -x\sin y + g'(y) = 3y^2 - x\sin y$, so that $g'(y) = 3y^2$. It follows that $g(y) = y^3$, so one choice is

$$f(x,y) = y^3 + x\cos y.$$

(b) Let C be the curve $\mathbf{r}(t) = \langle e^{t^2 \sin t}, t\cos^2 t\rangle$ for $0 \le t \le \pi$, and let $\mathbf{F}$ be the vector field from part (a) above. Use your answer from part (a) to find the value of $\int_C \mathbf{F}\cdot d\mathbf{r}$ without directly calculating the integral.

The fundamental theorem for line integrals says we can compute this integral by evaulating the potential function at the endpoints and subtracting. Note that the endpoints of C are $\mathbf{r}(\pi) = (1,\pi)$ and $\mathbf{r}(0) = (1,0)$. Thus we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = f(\mathbf{r}(\pi)) - f(\mathbf{r}(0)) = f(1,\pi) - f(1,0) = \pi^3 - 2.$$

4. Let C be the portion of the graph of $y = x^3$ where $0 \le x \le 1$, and let $\mathbf{F}(x,y) = \langle x^2 y, y\rangle$. Find $\int_C \mathbf{F}\cdot d\mathbf{r}$.

This is a line integral along a curve that is not closed, so we cannot use Green's theorem. The vector field is not conservative, so we cannot use the fundamental theorem of line integrals. Our only choice is to evaluate directly. To this end, we parametrize C via $\mathbf{r}(t) = \langle t, t^3\rangle$, so that $\mathbf{r}'(t) = \langle 1, 3t^2\rangle$. Also note that $\mathbf{F}(\mathbf{r}(t)) = \langle t^5, t^3\rangle$. Thus we have

$$\int_C \mathbf{F}\cdot d\mathbf{r} = \int_0^1 \mathbf{F}(\mathbf{r}(t))\cdot \mathbf{r}'(t)\,dt = \int_0^1 \langle t^5, t^3\rangle \cdot \langle 1, 3t^2\rangle\,dt = \int_0^1 4t^5\,dt = \frac{2}{3}.$$

5. (a) State the divergence form of Green's theorem.

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_D (\mathrm{div}\mathbf{F})\,dA$$

(b) Let R be the region in the upper half-plane $y \ge 0$ between the circles $x^2+y^2 = 1$ and $x^2+y^2 = 4$. Let C be the boundary of R, oriented counterclockwise. Let $\mathbf{F}(x,y) = \langle \cos(y^2) + 2x, e^x - y\rangle$. Use Green's theorem to find the normal component of $\mathbf{F}$ along C; i.e., find $\int_C \mathbf{F}\cdot\mathbf{n}\,ds$, where $\mathbf{n}$ is the outward pointing normal vector for C.

We use the theorem above. Note that $\mathrm{div}\mathbf{F} = 2 - 1 = 1$. Thus we have

$$\int_C \mathbf{F}\cdot\mathbf{n}\,ds = \iint_R \mathrm{div}\mathbf{F}\,dA = \iint_R dA = \mathrm{area}(R) = \frac{1}{2}(4\pi - \pi) = \frac{3\pi}{2}.$$

6. Let $\mathbf{F}(x,y,z) = \langle xy, y^2 - x^2, x^2z^2 \rangle$. Calculate $\operatorname{curl} \mathbf{F}$.

$$\operatorname{curl} \mathbf{F} = \langle 0 - 0, -(2xz^2 - 0), -2x - x \rangle = \langle 0, -2xz^2, -3x \rangle$$

7. (a) Express the area of the surface $\mathbf{r}(u,v) = \langle u\cos v, u\sin v, 2v \rangle$, with domain D given by $0 \le u \le 1$ and $0 \le v \le 2\pi$, as a double integral in u and v. Do not evaluate the integral.

We have that

$$\begin{aligned} |\mathbf{r}_u \times \mathbf{r}_v| &= |\langle \cos v, \sin v, 0 \rangle \times \langle -u\sin v, u\cos v, 2 \rangle| = |\langle 2\sin v, -2\cos v, u \rangle| \\ &= \sqrt{4\sin^2 v + 4\cos^2 v + u^2} = \sqrt{4 + u^2}. \end{aligned}$$

Thus the area is

$$\text{area}(S) = \iint_S dS = \iint_D |\mathbf{r}_u \times \mathbf{r}_v| \, dA = \int_0^{2\pi} \int_0^1 \sqrt{4 + u^2} \, du \, dv.$$

(b) Find the flux of the vector field $\mathbf{F}(x,y,z) = \langle 7x, y, 4z \rangle$ across the surface $\mathbf{r}(u,v) = \langle v^2, -uv, u^2 \rangle$, where the domain D in the uv-plane is the region with $0 \le u \le 1$ and $0 \le v \le 2u$.

We have that

$$\mathbf{r}_u \times \mathbf{r}_v = \langle 0, -v, 2u \rangle \times \langle 2v, -u, 0 \rangle = \langle 2u^2, 4uv, 2v^2 \rangle.$$

Also

$$\mathbf{F}(\mathbf{r}(u,v)) = \langle 7v^2, -uv, 4u^2 \rangle.$$

Thus

$$\mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) = 14u^2v^2 - 4u^2v^2 + 8u^2v^2 = 18u^2v^2.$$

We now calculate

$$\text{flux} = \iint_S \mathbf{F} \cdot d\mathbf{S} = \iint_D \mathbf{F}(\mathbf{r}(u,v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) \, dA = \int_0^1 \int_0^{2u} 18u^2v^2 \, dv \, du = \int_0^1 48u^5 \, du = 8.$$