1. Consider the following parametric curve:

$$x = t^2 \qquad y = t^3 - 3t.$$

(a) Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$.

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3t^2 - 3}{2t}$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}(dy/dx)}{dx/dt} = \frac{6t(2t) - 2(3t^2 - 3)}{(2t)^3} = \frac{3t^2 + 3}{4t^3}$$

(b) Find all values of t for which the tangent to the curve is horizontal and all values for which the tangent is vertical.

The tangent is horizontal when the slope is zero. Thus we find where $\frac{dy}{dx} = 0$. From above, this occurs when $3t^2 - 3 = 0$, or when $t = \pm 1$. The tangent is vertical when the slope is infinite. Thus we find where $\frac{dy}{dx}$ blows up. From above, this occurs where $2t = 0$, or when $t = 0$.

(c) Set up, but do not evaluate, an integral equal to the length of the piece of the curve joining $(0, 0)$ to $(4, 2)$.

The two endpoints correspond to t values $t = 0$ and $t = 2$. Thus we have

$$L = \int_0^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt = \int_0^2 \sqrt{(2t)^2 + (3t^2 - 3)^2}\, dt$$

2. (a) Find Cartesian coordinates for the point with polar coordinates $(2\sqrt{2}, 3\pi/4)$.

Using $x = r\cos\theta$ and $y = r\sin\theta$, we find that $x = -2$ and $y = 2$.

(b) Find two more sets of polar coordinates describing the point from part (a), at least one of which has $r < 0$.

We may always add or subtract 2π from the angle, obtaining, for example, $(2\sqrt{2}, 11\pi/4)$. Adding or subtracting π from the angle will change the sign of r, so we obtain $(-2\sqrt{2}, 7\pi/4)$, for example.

(c) Find the area inside the leminiscate $r^2 = 8\cos 2\theta$ and outside the circle $r = 2$.

To find the points of intersection, we set the expressions for r (or, in this case, r^2) equal to one another to obtain $8\cos 2\theta = 4$, or $\cos 2\theta = \frac{1}{2}$. This implies that $2\theta = \pi/3$, or $\theta = \pi/6$. Using the symmetries of the picture, we calculate

$$\text{Area} = 4\left[\int_0^{\pi/6} \frac{1}{2}(8\cos 2\theta)\,d\theta - \int_0^{\pi/6} \frac{1}{2}(2)^2\,d\theta\right]$$

$$16\left(\frac{\sin 2\theta}{2}\Big|_0^{\pi/6}\right) - 8\left(\theta\Big|_0^{\pi/6}\right) = 4\sqrt{3} - \frac{4\pi}{3}$$