Quiz 2 Solutions November 2, 2006

1. Find at least the first five non-zero terms of the Taylor series for $f(x) = \sqrt{x}$ centered at $x = 1$.

`We have that`

$$f(x) = x^{1/2} \Rightarrow f(0) = 1, \qquad f'(x) = \frac{1}{2}x^{-1/2} \Rightarrow f'(0) = \frac{1}{2},$$

$$f''(x) = -\frac{1}{4}x^{-3/2} \Rightarrow f''(0) = -\frac{1}{4}, \qquad f'''(x) = \frac{3}{8}x^{-5/2} \Rightarrow f'''(0) = \frac{3}{8},$$

$$f^{(4)}(x) = -\frac{15}{16}x^{-7/2} \Rightarrow f^{(4)}(0) = -\frac{15}{16},$$

`so that the Taylor series is`

$$1 + \frac{1/2}{1}(x-1) - \frac{1/4}{2}(x-1)^2 + \frac{3/8}{3!}(x-1)^3 - \frac{15/16}{4!}(x-1)^4 - \cdots .$$

2. Use known series to find the Taylor series (centered at zero) for the following function, finding at least five non-zero terms. Also find the radius of convergence:

$$\int x^3 e^{x^2}\, dx$$

`We know that`

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots ,$$

`so by substitution we have`

$$e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = 1 + x^2 + \frac{x^4}{2} + \frac{x^6}{3!} + \frac{x^8}{4!} + \cdots .$$

`Multiplying by` x^3 `yields`

$$x^3 e^{x^2} = \sum_{n=0}^{\infty} \frac{x^{2n+3}}{n!} = x^3 + x^5 + \frac{x^7}{2} + \frac{x^9}{3!} + \frac{x^{11}}{4!} + \cdots .$$

`Integrating then produces`

$$\int x^3 e^{x^2}\, dx = \sum_{n=0}^{\infty} \frac{x^{2n+4}}{(2n+4)n!} = \frac{x^4}{4} + \frac{x^6}{6} + \frac{x^8}{16} + \frac{x^{10}}{60} + \frac{x^{12}}{288} + \cdots .$$

3. Find the interval of convergence for the following series (don't forget the endpoints):

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n2^n} = \frac{x-2}{2} + \frac{(x-2)^2}{8} + \frac{(x-2)^3}{24} + \frac{(x-2)^4}{64} + \cdots .$$

Using the ratio test, we find that

$$\lim_{n\to\infty}\left|\frac{(x-2)^{n+1}n2^n}{(n+1)2^{n+1}(x-2)^n}\right| = \lim_{n\to\infty}\left[\frac{|x-2|}{2}\Big(\frac{n}{n+1}\Big)\right] = \frac{|x-2|}{2}.$$

Setting this less than 1 produces

$$\frac{|x-2|}{2} < 1 \quad \Rightarrow \quad |x-2| < 2 \quad \Rightarrow \quad -2 < x-2 < 2 \quad \Rightarrow \quad 0 < x < 4.$$

Checking the endpoints, for $x = 0$ we have

$$\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} = \ln(2),$$

while for $x = 4$ we have

$$\sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n} \to \infty.$$

Thus the interval of convergence is $[0,4)$.